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HIGH RESOLUTION IMAGING OF SPACE OBJECTS

1 OCTOBER 1979 TO 30 SEPTEMBER 1980

JAMES R. FIENUP
Radar and Optics Division

MARCH 1981

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This report describes the results of a research effort to investigate a method of obtaining high resolution images of space objects using earth-bound optical telescopes despite the turbulence of the atmosphere. The results of this research are an indication that using an iterative reconstruction algorithm, it is feasible to reconstruct diffraction-limited images from the Fourier modulus (or autocorrelation) data provided by stellar speckle interferometry.

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ABSTRACT (CONT'D)

Experiments were performed on astronomical data. It was necessary to develop methods for correcting for systematic errors and noise in the data. These methods were applied to [illegible] the data, and a diffraction-limited image was successfully reconstructed from the [illegible] data.

[illegible] images reconstructed from Fourier modulus data [illegible]. It was shown, among other things, that [illegible] separated parts, satisfying certain disconnection conditions, [illegible] uniquely specified by its Fourier modulus.

[illegible] was developed for reconstructing the support of an object from the [illegible] taking the intersection of three translates of the [illegible]. For objects consisting of a number of separated points, a [illegible] the object.

4

SUMMARY

This report describes the results of a research effort to investigate a method of obtaining high resolution images of space objects using earth-bound optical telescopes despite the turbulence of the atmosphere. The results of this research are an indication that, using an iterative reconstruction algorithm, it is feasible to reconstruct diffraction-limited images from the Fourier modulus (or autocorrelation) data provided by stellar speckle interferometry.

Experiments were performed on astronomical data. It was necessary to develop methods of compensating for systematic errors and noise in the data. These methods were applied to binary star data, and a diffraction-limited image was successfully reconstructed from the resulting Fourier modulus data.

The uniqueness of images reconstructed from Fourier modulus data was explored using the theory of analytic functions. It was shown, among other things, that if an object or its autocorrelation consists of separated parts satisfying certain disconnection conditions, then it is usually uniquely specified by its Fourier modulus.

A new method was developed for reconstructing the support of an object from the support of its autocorrelation; it involves taking the intersection of three translates of the autocorrelation support. For objects consisting of a number of separated points, a new method was developed for reconstructing the object.

FOREWORD

This report was prepared by the Radar and Optics Division of the Environmental Research Institute of Michigan. The work was sponsored by the Air Force Office of Scientific Research/AFSC, United States Air Force, under Contract No. F49620-80-C-0006.

This interim scientific report covers work performed between 1 October 1979 and 30 September 1980. The contract monitor is Dr. Henry Radoski, Directorate of Physical and Geophysical Sciences, AFOSR/NP, Building 410, Bolling Air Force Base, D.C. 20332. The principal investigator is James R. Fienup. Major contributors to the effort are Thomas R. Crimmins and James R. Fienup. Additional contributors to the effort are Gerald B. Feldkamp, Lawrence S. Joyce, Emmett N. Leith, and Christopher J. Roussi.

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HIGH RESOLUTION IMAGING OF SPACE OBJECTS

1
INTRODUCTION AND OBJECTIVES

This report describes the results of the first year of a two-year research effort to investigate a method of obtaining high resolution images of space objects using earth-bound optical telescopes.

A serious problem in astronomy is that the turbulence of the earth's atmosphere severely limits the resolution of large earth-bound optical telescopes. Under good "seeing" conditions the resolution allowed by the atmosphere is typically one second of arc, compared with 0.02 seconds of arc, the theoretical diffraction-limited resolution of a five-meter diameter telescope. That is, the potential exists for obtaining images having fifty times finer resolution than what is ordinarily obtainable.

Several interferometric methods are capable of providing high-resolution (diffraction-limited) information through atmospheric turbulence. The most promising of these interferometric methods is Labeyrie's stellar speckle interferometry. The high-resolution information provided by these methods is the modulus of the Fourier transform of the object; the phase of the Fourier transform is lost. Unfortunately, except for the very special case in which an unresolved star is very near the object of interest, the Fourier modulus can be used to directly compute only the autocorrelation of the object, but not the object itself. The autocorrelation is ordinarily useful only for determining the diameter of the object or the separation of a binary star pair.

In recent years it has been shown that this stumbling block can be overcome by an iterative algorithm for computing the object's spatial (or angular) brightness distribution from its Fourier modulus. The algorithm relies both on the Fourier modulus data measured by stellar speckle interferometry and on the a priori constraint that

the object distribution is a nonnegative function. Therefore, the combination of stellar speckle interferometry with the iterative algorithm can provide diffraction-limited images despite the presence of atmospheric turbulence.

The goals of this two-year research effort are threefold: (1) to improve the iterative reconstruction algorithm to make it operate reliably in near real-time on imperfect real-world data, (2) to determine the uniqueness of the solution under various conditions, and (3) to demonstrate the reconstruction technique with real-world interferometer data, thereby providing images with finer resolution than would ordinarily be possible.

As envisioned in the statement of work for this contract, these goals would be met as follows:

A. Perform initial studies and set priorities for the following five study areas:

   1. analytical study of the input-output concept using a statistical approach.

   2. analytical and computer studies of the uniqueness problem.

   3. variations of the basic algorithm to improve reliability.

   4. analysis and computer simulations of the effects of noise and other imperfections in the data, and methods for minimizing their effects for the types of noise present in conventional interferometers.

   5. combining the iterative approach with other imaging techniques such as the Knox-Thompson method.

B. Perform detailed studies of those areas listed above that are found to be most important.

C. Obtain interferometer data, evaluate it, and process it into imagery.

D. Study the applicability of the iterative technique to other problems.

2
RESEARCH ACCOMPLISHMENTS

The first year's research effort can be divided into three major topics. (1) Stellar speckle interferometer data was acquired, evaluated, and processed into imagery. Methods were developed for minimizing the effects of the types of noise and imperfections found in that data[1]. (2) Analytical (and to a lesser extent computer) studies of the uniqueness problem were performed[2,3]. (3) A new method, not envisioned at the beginning of this program, was developed for reconstructing the support of an object; and for objects consisting of a number of point-like sources, a new noniterative method was developed for reconstructing the object[4]. Publications arising from this research effort are listed as References 1-4 below.

The results obtained for each of the three topics listed above are summarized in the three respective sections that follow. Reference 1 to 4 are included as Appendices A, B, C, and D, respectively. Chapter 3 of this report contains a more complete discussion of the support reconstruction method.

1. J.R. Fienup and G.B. Feldkamp, "Astronomical Imaging by Processing Stellar Speckle Interferometry Data," presented at the 24th Annual Technical Symposium of the SPIE, San Diego, Calif., 30 July 1980; and published in SPIE Proceedings Vol. 243, Applications of Speckle Phenomena (July 1980), p. 95.
2. T.R. Crimmins and J.R. Fienup, "Phase Retrieval for Functions with Disconnected Support," submitted to J. Math Physics.
3. T.R. Crimmins and J.R. Fienup, "Comments on Claims Concerning the Uniqueness of Solutions to the Phase Retrieval Problem," submitted to J. Opt. Soc. Am.
4. J.R. Fienup and T.R. Crimmins, "Determining the Support of an Object from the Support of Its Autocorrelation," presented at the 1980 Annual Meeting of the Optical Society of America, Chicago, Ill., 15 October 1980; Abstract: J. Opt. Soc. Am. **70**, 1581 (1980).

2.1 ASTRONOMICAL DATA PROCESSING

Stellar speckle interferometry data was obtained both from the Steward Observatory Stellar Speckle Interferometry Program (via K. Hege, Steward Observatory) and from the Anglo-Australian Telescope (via J.C. Dainty, U. Rochester).

The data from the Anglo-Australian Telescope was in the form of many short-exposure images on 16 mm cine film. A number of methods for digitizing the data were explored, and the one chosen, the most economical by far, was the following. The 16 mm film was cut into strips and contact copied, along with grey-scale step wedges, onto 9 x 9 inch sheets of film. The 9-inch sheets of film were then sent to the Image Processing Institute at the University of Southern California for digitization on their Optronics digitizer. Software would then have to be developed in order to extract the desired data from the digitized array (which includes 16 mm film sprocket holes, etc.). After the data was digitized it was discovered that the Optronics digitizer had been malfunctioning and required repairs. The film will have to be redigitized before further experimentation can proceed with this data.

Considerable progress was made with the Steward Observatory data, which was already in digital form. A description of that work is found in Appendix A (Ref. 1), and is summarized below.

It was previously known that it is necessary to compensate the Fourier modulus data for a certain noise bias term due to photon noise. Using the Steward Observatory data, it was found that the detection process resulted in a frequency transfer function, which we call the detection transfer function, which, in addition to being an error itself, prevented the compensation of the noise bias. Methods of determining the detection transfer function from the data and compensating for it was developed. Methods of compensating for other systematic errors were also developed. These methods were

applied to stellar speckle interferometry data of a binary star system, and a diffraction-limited image was successfully reconstructed from the resulting compensated Fourier modulus data.

Having gained this experience with single and binary star data, the next step will be to use the same methods on more complicated objects, such as asteroids or Jovian moons.

2.2 UNIQUENESS THEORY

The principle means of exploring the uniqueness of images reconstructed from Fourier modulus data has been the theory of analytic functions. As described in more detail in Appendix B, for the one-dimensional case there are usually many different objects having the same Fourier modulus. Examples of both uniqueness and non-uniqueness are given. However, it is shown that if a function or its autocorrelation satisfy certain disconnection conditions, then the solution is unique unless the separated parts of the function are related to one another in a special way. Therefore, a functions satisfying these conditions can usually be uniquely reconstructed from its Fourier modulus (or from its autocorrelation). It is also shown that if the non-real complex zeroes of the Fourier transform of a function of disconnected support are finite in number, then the support of the function as well as the function itself satisfy some special conditions. This makes it unlikely that the Fourier transform of a given function would have only a finite number of non-real zeroes.

In the course of this work it was discovered that some of the theory appearing previously in the literature was in error, as described in Appendix C (Ref. 3). The detailed corrected theory is contained in Appendix B.

To date the theory of analytic functions has not been extended to two dimensions, the case of most interest in this research effort. The two dimensional case is not a direct extension of the one-dimensional analysis. The high probability of ambiguous solutions

in one dimension does not seem to be the case in two dimensions. In a one-dimensional computer experiment using the iterative reconstruction algorithm on a case known to have two solutions (Figure 1 of Appendix B), the algorithm converged to one of the solutions in about half of the trials and converged to the other solution in the other half of the trials, depending on the random number sequence used as the initial input to the algorithm. Therefore, it is believed that if there are multiple solutions, then the algorithm is likely to find any one of them. For the case of complicated two-dimensional objects on the other hand, the algorithm generally converges to the object itself, and not to other solutions. This is an indication that other solutions do not usually exist in the two-dimensional case.

2.3 NEW METHODS FOR SUPPORT AND OBJECT RECONSTRUCTION

As discussed in more detail in Chapter 3 of this report (see also Appendix D, Ref. 4), a new method was developed for reconstructing the support of an object (the set of points at which it is nonzero) from the support of its autocorrelation. In some instances, for example to find the relative locations of a collection of point-like stars, the object's support is the desired information. More generally, once the object's support is known, then the complete reconstruction of the object by the iterative method is simplified. Several methods are shown of finding sets which contain all possible support solutions. Particularly small and informative sets containing the solutions are given by the intersections of two translates of the autocorrelation support. For the special case of convex objects, the intersection of three translates of the autocorrelation support generates a family of solutions to the support of the object. For the special case of an object consisting of a collection of points satisfying certain nonredundancy conditions, the intersection of three translates of the autocorrelation support generates a unique

solution. In addition, for these same objects, by taking the product of three translates of the autocorrelation function, one can reconstruct the object itself in addition to reconstructing the support of the object.

2.4 CONCLUSIONS

All of the results noted above are encouraging and are further indications that high resolution imaging by combining the iterative algorithm with stellar interferometry data is feasible. The preliminary experience with astronomical data shows that although additional problems exist with real-world data, the problems encountered so far can be overcome, and it is possible to reconstruct high-resolution images from such data. Fears that the Fourier modulus data might admit to multiple image solutions are largely unjustified. The theory of analytic functions predicts that a large class of one-dimensional functions are uniquely specified by their Fourier modulus; in addition, for the more practical two-dimensional case it appears that the vast majority of functions are uniquely determined by the Fourier modulus. Finally, new methods were developed for reconstructing the object's support from its autocorrelation's support, and even for reconstructing the object itself by a very simple method for the case of a collection of point-like stars.

3
RECONSTRUCTION OF THE SUPPORT OF AN OBJECT FROM THE SUPPORT OF ITS AUTOCORRELATION

3.1 INTRODUCTION

In astronomy, X-ray crystallography and other disciplines one often wishes to reconstruct an object from its autocorrelation or, equivalently, from the modulus of its Fourier transform (i.e., the phase retrieval problem)[5]. It is also useful to be able to reconstruct just the support of the object (the set of points over which it is nonzero). In some cases, for example, to find the relative locations of a number of point-like stars, the object's support is the desired information. In addition, once the object's support is known, the reconstruction of the object by the iterative method[6] is simplified. Therefore, we are motivated to find a quick way to determine the support of the object from the support of its autocorrelation.

In the general case there may be many solutions for the object's support given the autocorrelation support. In what follows a method for generating sets containing all possible solutions is given. In addition, for the special case of convex sets a method for generating a family of support solutions is described. For the special case of point-like objects this method is shown to yield a unique support solution unless the vector separations of the points in the object satisfy certain redundancy-type conditions. If instead of manipulating the autocorrelation support one uses the autocorrelation function, then for the same point-like objects one can reconstruct the object itself. In the following, several lengthy proofs are omitted for the sake of brevity.

5. See, for example, H.P. Baltes, ed., Inverse Source Problems in Optics (Springer-Verlag, New York, 1978).

6. J.R. Fienup, "Reconstruction of an Object from the Modulus of Its Fourier Transform," Opt. Lett. 3, 27 (1978); J.R. Fienup, "Space Object Imaging Through the Turbulent Atmosphere," Opt. Eng. 18, 529 (1979).

3.2 DEFINITIONS AND BACKGROUND

The results shown here apply to functions of any number of dimensions except where otherwise noted. For simplicity we consider only real, nonnegative functions. A function $f(x) \geq 0$, where $x \in E^N$, has <u>support</u> S, where S is the smallest closed set outside of which the function is zero almost everywhere.

The <u>autocorrelation of f(x)</u> is

$$f \bigstar f(x) = \int \ldots \int_{-\infty}^{\infty} f(y)\, f(y + x)\, dV(y) \tag{1}$$

$$= \int \ldots \int_{-\infty}^{\infty} f(y)\, f(y - x)\, dV(y) \tag{2}$$

where V is the volume measure on E^N. The Fourier transform of the autocorrelation of f(x) is equal to the squared modulus of the Fourier transform of f(x). Note that the autocorrelation is (centro-) symmetric. It is most illuminating to interpret Eq. (2) as a weighted sum of translated versions of f(x). That is, in the integrand of Eq. (2a), f(y) acts as the weighting factor for f(y + x), which is f(x) translated by -y. It can be shown that the support of the autocorrelation of f(x) is

$$A = \bigcup_{y \in S} (S - y)$$

$$= S - S = \{x - y:\ x,\ y \in S\} \tag{3}$$

Note that A is <u>symmetric</u>:

$$-A = A \tag{4}$$

To illustrate the interpretation of an autocorrelation support, consider the case of the two-dimensional support S shown in Figure 1(a), having the form of a triangle with vertices at points a, b, and c. The autocorrelation support A can be thought of as being formed by successively translating S so that each point in S is at the origin, and taking the union of all these translates of S. Figure 1(b) shows three such translates, (S - a),(S - b), and (S - c). The rest

of A is filled in, as shown in Figure 1(c), by including all $(S - y)$ such that $y \in S$.

We are concerned with the following problem. Given a symmetric set $A \subseteq E^N$ find sets $S \subseteq E^N$ which satisfy $A = S - S$.

Sets S_1 and S_2 are equivalent,

$$S_1 \sim S_2 \tag{5a}$$

if there exists a point v such that

$$S_2 = v + \beta S_1 \tag{5b}$$

$$= \{v + \beta x : x \in S_1\} \tag{5c}$$

where $\beta = + 1$ or $- 1$. From Eq. 3(b) it is easily seen that if S_1 is a solution to $S - S = A$, and if $S_2 \sim S_1$, then S_2 is also a solution. If S_1 is a solution and all other solutions are of the form $v + \beta S_1$, then the solution is said to be unique and A is said to be unambiguous; if there exist any nonequivalent solutions, then the solution is nonunique and A is ambiguous.

Not all symmetric sets are necessarily autocorrelation supports. For non-null sets A it follows from Eq. (3) that

$$0 \in A \tag{6}$$

is a necessary condition for the existence of a solution. The following example shows that this is not a sufficient condition. As shown in Figure 2, let $A = \{(0, 0), (1, 0), (-1, 0), (0, 1), (0, -1)\}$. Because of the points (0, 0), (1, 0), (-1, 0) a solution must include two points separated by (1, 0). Similarly because of the points (0, 0), (0, 1), (0, -1) a solution must include two points separated by (0, 1). Therefore, the solution must have at least three distinct noncolinear points. Of the three possible pairings of the three points, one has a separation along (1, 0), a second has a separation along (0, 1), and the third pair of points must have a diagonal separation. However, no diagonal terms appear in A, and therefore there is no solution for $A = S - S$ in this case.

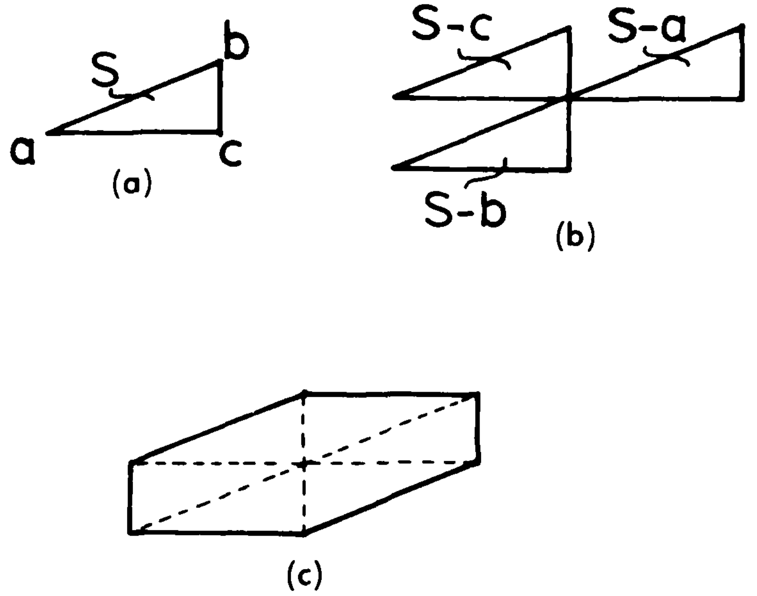

Figure 1. Autocorrelation Support. (a). Set S; (b) three of the translates of S that make up A; (c) autocorrelation support A = S-S.

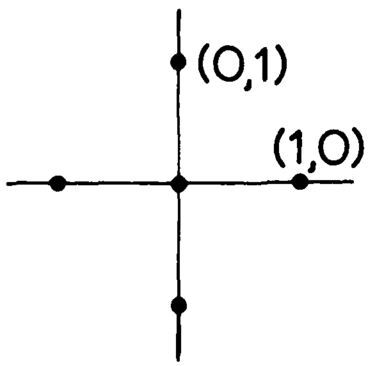

Figure 2. A Symmetric Set that Is Not an Autocorrelation Support

A set X is <u>convex</u> if for all $x, y \in X$,

$$tx + (1 - t)y \in X \tag{7}$$

for all $t \in [0, 1]$. The <u>convex hull</u> of a set X, denoted by c.hull(X) is given by the smallest convex subset of E^N containing X. Thus X is convex if and only if X = c.hull(X). If S is convex, then $A = S - S$ is also convex. More generally,

$$\text{c.hull}(X - X) = \text{c.hull}(X) - \text{c.hull}(X) \tag{8}$$

All convex symmetric sets A have at least one solution

$$S = \frac{1}{2} A = \{x/2 : x \in A\} \tag{9}$$

The proof is as follows. Let $u, v \in 1/2\ A$. Then $2u \in A$, $2v \in A$ and $-2v \in A$. Therefore,

$$u - v = \frac{1}{2}(2u) + \frac{1}{2}(-2v) \in A \tag{10}$$

and so $(1/2\ A) - (1/2\ A) \subseteq A$. Now let $v \in A$. Then $v/2 \in 1/2\ A$ and $-v/2 \in 1/2\ A$. Therefore

$$v = (\tfrac{1}{2}v) - (-\tfrac{1}{2}v) \in (\tfrac{1}{2}A) - (\tfrac{1}{2}A) \tag{11}$$

and so $A \subseteq (1/2\ A) - (1/2\ A)$. Therefore

$$A = (\tfrac{1}{2}A) - (\tfrac{1}{2}A) \tag{12}$$

3.3 LOCATOR SETS

In many cases A is ambiguous, and it would be useful to define a set that contains all possible solutions. A set $L \subseteq E^N$ is defined as a <u>locator set</u> for A if for every closed set $S \subseteq E^N$ satisfying $A = S - S$, some translate of S is a subset of L, i.e., there exists a vector v such that

$$v + S \subseteq L \tag{13}$$

There are many ways to generate locator sets. For example, for $v \in S$, $S - v \subset S - S = A$, and so A itself is a locator set. Naturally, the smaller the locator set, the more tightly it bounds the possible solutions, and the more informative it is. It can be shown that a smaller locator set than A is

$$L = A \cap H \tag{14}$$

where H is a closed half-space with the origin on its boundary. A still smaller locator set can be shown to be

$$L = \frac{1}{2} P \tag{15}$$

where P is any N-dimensional parallelopiped containing A.

A particularly interesting locator set is given by the following intersection of two autocorrelation supports. If $w \in A$, then

$$L = A \cap (w + A) \tag{16}$$

is a locator set for A. Note that L is symmetric about the point $w/2$. The proof that this is a locator set is as follows. Suppose S satisfies $A = S - S$. Since $w \in A$, there exist $u, v \in S$ such that $w = u - v$. Consider $z \in S - v$. Then $z = s - v$ where $s \in S$, $z = s - v \in A$, and $z = s - u + (u - v) = s - u + w \in A + w$. Therefore, $z \in A \cap (A + w) = L$ and $S - v \subset L$.

Naturally, the most interesting (smallest) locator sets generated by this method of intersecting two autocorrelation supports are obtained by choosing w to be on the boundary of A. By choosing different points $w \in A$, a whole family of locator sets can be generated by this method.

Example 1.

Consider the set S shown in Figure 3(a), consisting of two balls joined by two thin rods, and its autocorrelation support $A = S - S$ shown in Figure 3(b). An example of a locator set 1/2 P is shown in

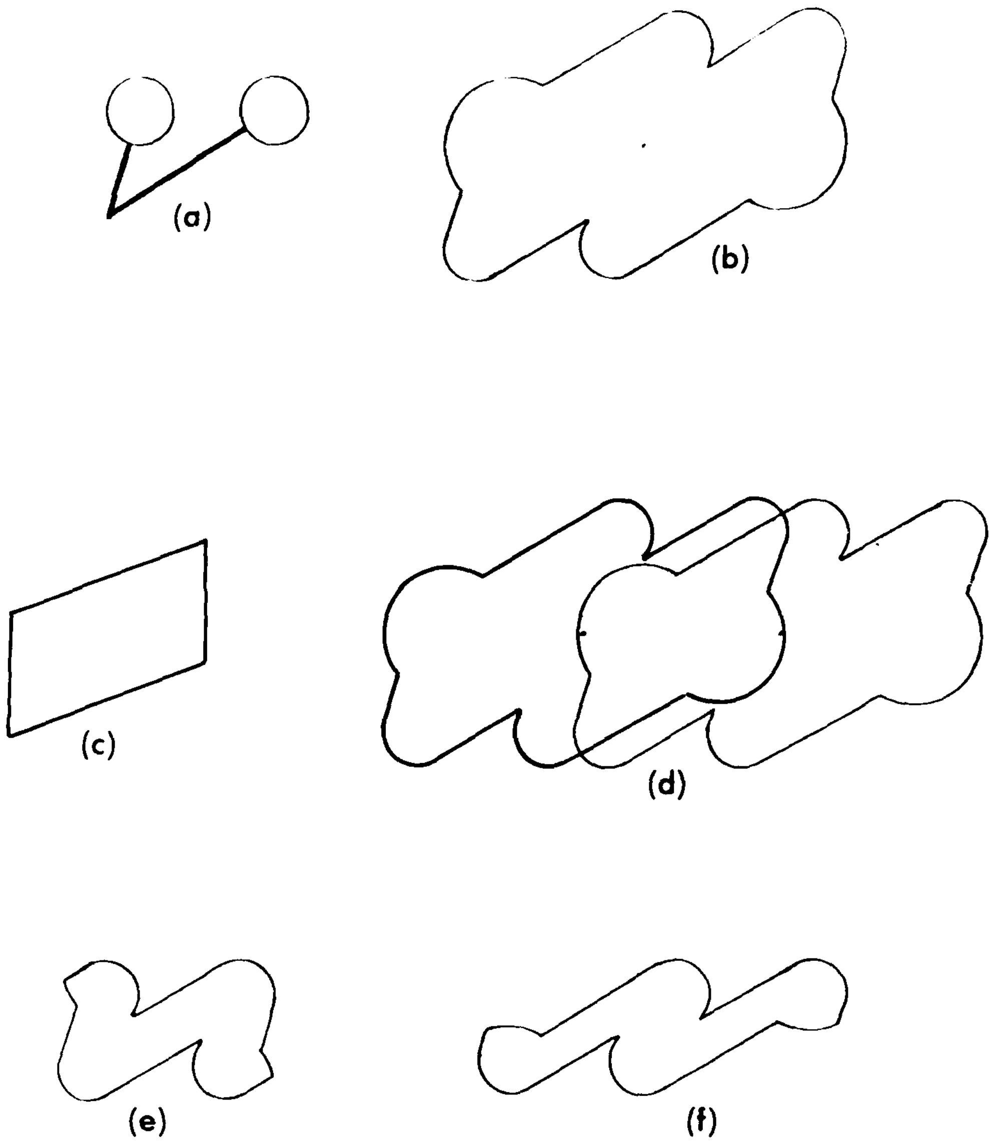

Figure 3. Locator Sets. (a) Set S; (b) $A = S-S$; (c) locator set $L = 1/2\ P$; (d) formation of $L = A \cap (w + A)$; (e) and (f) two other members of the family of locator sets.

Figure 3(c); it does a good job of defining the approximate size of S, but it is not suggestive of any of the details of its shape. Figure 3(d) shows the generation of the locator set $L = A \cap (w + A)$ for a particular point $w \in A$. Figure 3(e) and 3(f) show two other members of the family of locator sets generated with two other points $w \in A$. These locator sets generated by intersecting two autocorrelation supports are very suggestive of the shape of the solution (or solutions). This is especially true if one realizes that any solution must be contained within all of these locator sets. Unfortunately, for the general case it is difficult to narrow down the solution any further: a way to combine the information from two or more of the family of locator sets has not been devised. However, as will be shown in the sections that follow, for special classes of sets much more can be done.

3.4 AUTOCORRELATION TRI-INTERSECTION FOR CONVEX SETS

For the special case of convex sets A, a family of solutions can be generated by a simple method. For the one-dimensional case the result is trivial: a unique solution is given by $S = 1/2\ A$, which is just a segment of the line half the length of the line segment A. An equivalent result for the one-dimensional convex case is the solution

$$S = A \cap (w + A) \tag{17}$$

where w is on the boundary of A (at one end of the line segment A), or in symbols $w \in \partial(A)$.

For the two-dimensional convex case, we have the following result. Let $A \subseteq E^2$ be a closed convex symmetric set with non-null interior, and let

$$w_1 \in \partial(A) \text{ and } w_2 \in \partial(A) \cap \partial(w_1 + A).$$

Furthermore, let

$$B = A \cap (w_1 + A) \cap (w_2 + A). \tag{19}$$

Then

$$A = B - B. \tag{20}$$

The lengthy proof of this result is omitted for the sake of brevity.

Example 2

Consider the set S shown in Figure 4(a), which is the convex hull of the set shown in Figure 3(a). Its autocorrelation support $A = S - S$ [which is the convex hull of Figure 3(b)] is shown in Figure 4(b). The parallelogram shown in Figure 3(c) is a locator set for A. A member of the family of locator sets $A \cap (w + A)$ is shown by the intersection of A and $w + A$ in Figure 4(c). A member of the family of solutions B is shown by the intersection of the three sets $A \cap (w_1 + A) \cap (w_2 + A)$ in Figure 4(d). Two other examples of B are shown in Figures 4(e) and 4(f).

3.5 THREE-DIMENSIONAL INTERSECTIONS OF CONVEX SETS

For convex sets, since in one dimension the intersection of two sets, Eq. (17), results in the solution, and since in two dimensions the intersection of three sets, Eq. (19), results in solutions, one might hope that in three dimensions the set

$$C = A \cap (w_1 + A) \cap (w_2 + A) \cap (w_3 + A) \tag{21}$$

would be a solution to $S - S = A$, where $w_1 \in \partial(A)$, $w_2 \in \partial(A) \cap \partial(w_1 + A)$, and $w_3 \in \partial(A) \cap \partial(w_1 + A) \cap \partial(w_2 + A)$. Unfortunately, this is generally not the case.

A counter-example to $C - C = A$ is the following. Consider S equal to a sphere of diameter one, then $A = S - S$ is a sphere of radius one centered at the origin. Figures 5(a) and 5(b) show

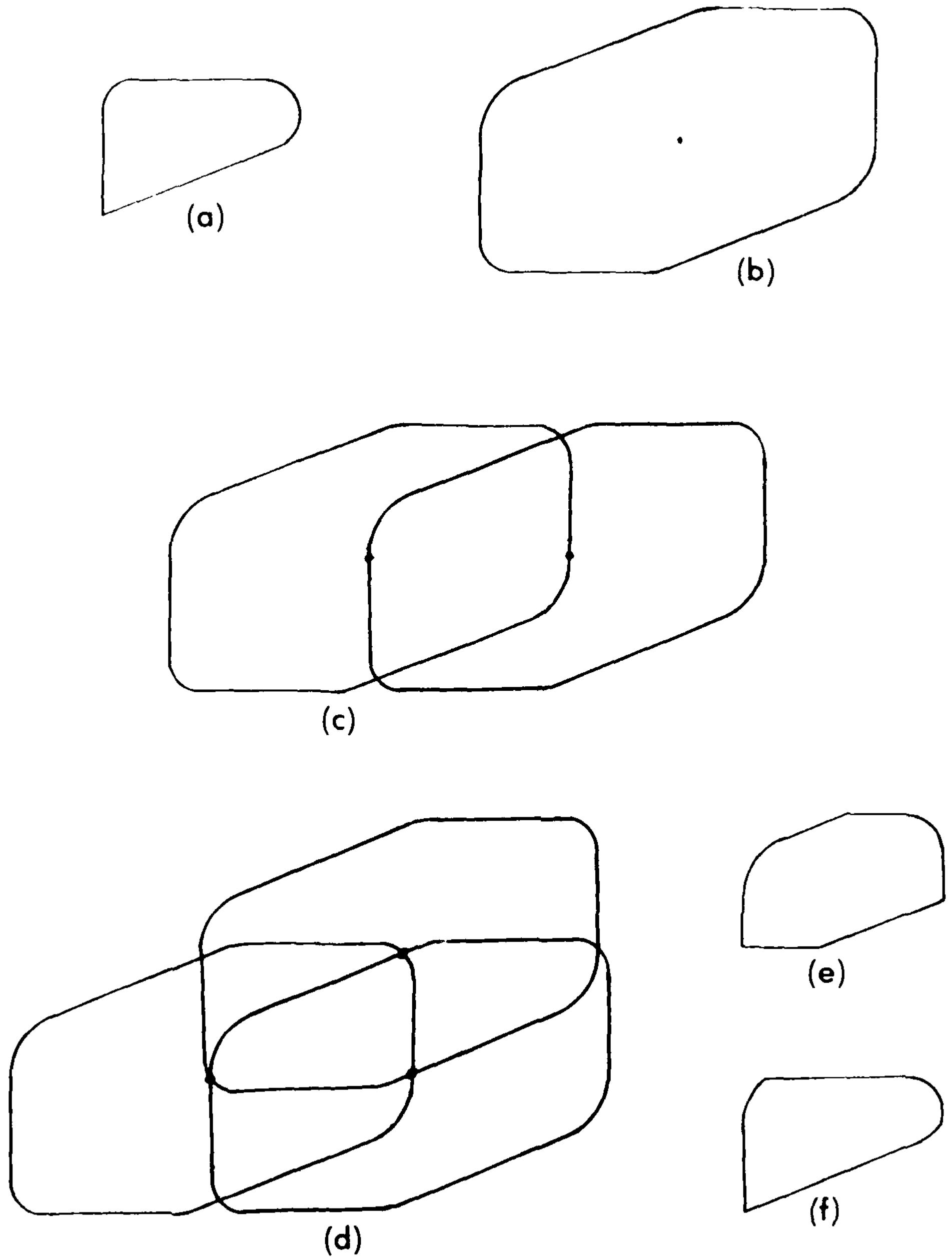

Figure 4. Autocorrelation Tri-Intersection Solutions for Convex Sets. (a) Set S; (b) $A = S-S$; (c) formation of locator set $L = A \cap (w + A)$; (d) formation of solution $B = A \cap (w_1 + A) \cap (w_2 + A)$; (e) and (f) two other solutions of the form B.

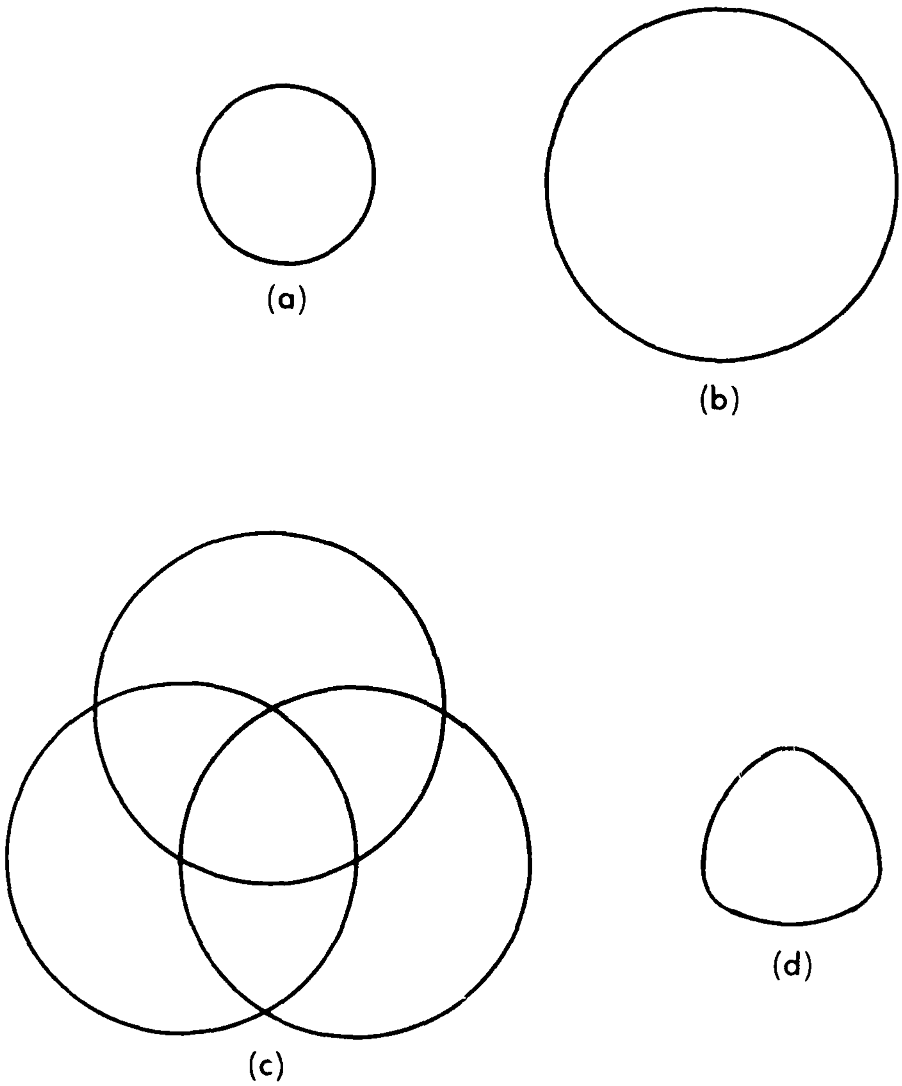

Figure 5. Sphere/Circle Example. (a) Set S; (b) $A = S - S$; (c) $B = A \cap (w_1 + A) \cap (w_2 + A)$; (d) another solution to the circle.

planar cuts through the centers of S and A, respectively. Figure 5(c) shows a planar cut through $A \cap (w_1 + A) \cap (w_2 + A)$ through the three points, 0, w_1 and w_2. $A \cap (w_1 + A) \cap (w_2 + A)$ has two vertices, one in front of the plane of the page and one behind the plane of the page, both at distance one from the centers of each of the three intersecting spheres. Taking the intersection of this with $(w_3 + A)$, which is centered at one of the two vertices, gives us C, which is similar to a regular tetrahedron (it has the same vertices) but having spherical surfaces of radius one in place of the four plane faces of a tetrahedron. Looking for a moment at the tetrahedron T having the same vertices as C (i.e., the convex hull of points 0, w_1, w_2, and w_3 having edges of length one), we see that $T - T$ is a cuboctahedron, which has eight triangular faces and six square faces. Since $T \subset C$, $T - T \subset C - C$. The surface of $C - C$ can be subdivided into twelve patches associated with the twelve faces of the cuboctahedron. It can be shown that the eight patches associated with the triangular faces coincide exactly with the surface of the sphere A of radius one. However, the six patches corresponding to the square faces do not. For example, the distance from the origin to the center of each of those six patches is equal to the distance between the centers of two opposing edges of C. This distance can be shown to be $\sqrt{3} - \sqrt{2}/2 \approx 1.0249$; that is, $C - C$ bulges beyond the sphere by about 2.49 percent at those points. Therefore, $C - C \neq A$.

3.6 COMBINATIONS OF CONVEX SOLUTIONS

Solutions to convex $A = S - S$ of the form B make up a family of solutions generally having an uncountable infinity of members, one for each $w_1 \in \partial(A)$. Nevertheless, there may exist additional solutions.

Additional solutions can be generated in the following way. If S_1 and S_2 are solutions to convex $A = S - S$, then

$$S_t = tS_1 + (1 - t)S_2 \tag{22}$$

is also a solution for $0 \leq t \leq 1$. The proof of this result is as follows

$$\begin{aligned} S_t - S_t &= [tS_1 + (1 - t)S_2] - [tS_1 + (1 - t)S_2] \\ &= tS_1 - tS_1 + (1 - t)S_2 - (1 - t)S_2 \\ &= tA + (1 - t)A \\ &= A \end{aligned} \tag{23}$$

since A is convex.

If S_1 is a solution, then so is $- S_1$. Then using $t = 1/2$ and $S_2 = - S_1$ in Eq. (22), it is seen that

$$S_{1/2} = \frac{1}{2} S_1 - \frac{1}{2} S_1 = \frac{1}{2} A \tag{24}$$

is a solution, as was previously shown by Eq. (12).

Eq. (22) can easily be generalized as follows. If $S_1, \ldots, S_n$ are solutions for convex A, and if $t_1, \ldots, t_n \geq 0$ and $t_1 + t_2 + \ldots + t_n = 1$, then

$$S = \sum_{i=1}^{n} t_i S_i \tag{25}$$

is also a solution.

Example 3

Consider the two-dimensional convex set S shown in Figure 5(a), consisting of a circle of diameter one. $A = S - S$, consisting of a circle of radius one is shown in Figure 5(b), and a tri-intersection solution B is shown as the intersection of three circles in Figure

5(c). This solution is similar to an equilateral triangle but having arcs of circles of radius one with centers at the opposite vertices for each of the three sides. It can easily be seen that all other solutions B generated by Eq. (19) are similar to the one shown in Figure 5(c) except rotated in the plane. The circle of diameter one shown in Figure 5(a) is not of this form, but it is also a solution to A. As shown by Eq. (24), $S = 1/2\ A$ in Figure 5(a) can be generated by applying Eq. (22), using $S_1 = -S_2 = B$ and $t = 1/2$. One of a family of additional solutions generated by Eq. (22) is shown in Figure 5(d). It was generated using $S_1 = 1/2\ A$ in Figure 5(a), $S_2 = B$ in Figure 5(c), and $t = 1/2$.

3.7 THE AMBIGUITY OF CONVEX SETS

We now consider the question of uniqueness of convex solutions of $A = S - S$ for convex A. As mentioned earlier, $1/2\ A$ is a solution. If all convex solutions are equivalent to S, then A is said to be convex-unambiguous. It was shown that in two dimensions one can generate a family of solutions by Eq. (19), the member of the family being determined by the choice of w_1. Eq. (22) or (25) can then be used to generate still more solutions. Therefore one would suppose that convex sets A are generally convex-ambiguous. However, it is also possible that all solutions generated by Eq. (19) are equivalent, in which case A would be convex-unambiguous.

In what follows it is shown that in two dimensions if A is a parallelogram then A is convex-unambiguous. Let A be a parallelogram having vertices w_1, $-w_1$, w_2, and $-w_2$. By Eq. (16) a locator set for A is $L = A \cap (w_1 + A)$ since $w_1 \in A$. It is easily seen that $L = 1/2\ w_1 + 1/2\ A$, and so $L' = 1/2\ A$, which has vertices $1/2\ w_1$, $-1/2\ w_1$, $1/2\ w_2$, $-1/2\ w_2$ is a locator set for A. Suppose $A = S - S$ where S is convex. Then some translate of S, call it S', is contained in L'. Since $w_1 \in A$ there exist u, v $\in$ S' such that $w_1 = u - v$. Since $S' \subseteq L'$, u, v $\in$ L'. It follows

that $u = 1/2\, w_1$ and $v = -1/2\, w_1$. Therefore, $1/2\, w_1 \in S'$ and $-1/2\, w_1 \in S'$. Similarly $1/2\, w_2 \in S'$ and $-1/2\, w_2 \in S'$. Then, since S' is convex

$$L' = \text{c.hull}\left[\left\{\frac{1}{2}w_1, -\frac{1}{2}w_1, \frac{1}{2}w_2, -\frac{1}{2}w_2\right\}\right] \subseteq S' \subseteq L' \qquad (26)$$

Therefore, $S = L' = 1/2\, A$, and so S is unique among convex solutions.

It can also be shown that parallelograms are the <u>only</u> two-dimensional convex-unambiguous sets. Convex symmetric sets $A \subseteq E^2$ that are not parallelograms can be shown to have infinitely many nonequivalent solutions to $A = S - S$.

3.8 AUTOCORRELATION TRI-INTERSECTION FOR POINT-LIKE SETS

For the special case of certain point-like sets A, the solution can be generated by a method similar to the one for convex sets. By point-like sets we mean sets comprised of a collection of distinct noncontiguous points. For example, a point-like function

$$f(x) = \sum_{n=1}^{N} f_n\, \delta(x - x_n) \qquad (27)$$

consisting of N delta functions having amplitudes $f_n > 0$, $n = 1, \ldots, N$, would have point-like support

$$S = \left\{x_n : n = 1, \ldots, N\right\}. \qquad (28)$$

The following result holds for any number of dimensions. Let S be a point-like set and $A = S - S$. Let $w_1 \in A$ and $w_2 \in A \cap (w_1 + A)$, with $0 \neq w_1 \neq w_2 \neq 0$, and let

$$B = A \cap (w_1 + A) \cap (w_2 + A) \qquad (29)$$

Define the following Condition 1:

If $x_1, x_2, y_1, y_2, z_1, z_2 \in S$, $x_1 \neq x_2$, and

$$x_1 - x_2 + y_1 - y_2 + z_1 - z_2 = 0 \qquad (30)$$

then

$$x_1 = y_2 \text{ or } z_2, \quad \text{and } x_2 = y_1 \text{ or } z_1$$

It can be shown that if S satisfies Condition 1, then

$$B \sim S \qquad (31)$$

That is, B is the unique solution to A = S - S.

Another approach is as follows. Define Condition 2: if the set $G \subseteq A$ consists of three distinct points and if $0 \in G$ and $G - G \subseteq A$, then G is equivalent to a subset of S. Define Condition 3:

if $x_1, x_2, y_2 \in S$, $x_1 \neq x_2$, and $x_1 - x_2 = y_1 - y_2$; then $x_1 = y_1$.

We have the following two results. If S satisfies Condition 2, then S is equivalent to a subset of B. It can also be shown that if S satisfies Conditions 2 and 3, then S is equivalent to B; and S satisfies Conditions 2 and 3 if and only if it satisfies Condition 1.

Since it requires a special relationship between the points in S in order that Condition 1 not be satisfied, it is probable that for S comprised of randomly located points, B is the unique solution to A = S - S. More will be said about this later.

Example 4

Consider the point-like set S having 9 points shown in Figure 6(a). A = S - S shown in Figure 6(b) has $9^2 - 9 + 1 = 73$ points. Intersecting A with a translate of itself using Eq. (16), a number of different locator sets for A can be formed, two of which are shown in Figures 6(c) and (d). (Any solution to A = S - S must have translates that are subsets of all the locator sets.) For this example, for all allowable values of w_1 and w_2, B is found to be equivalent to S, which is shown in Figure 6(a).

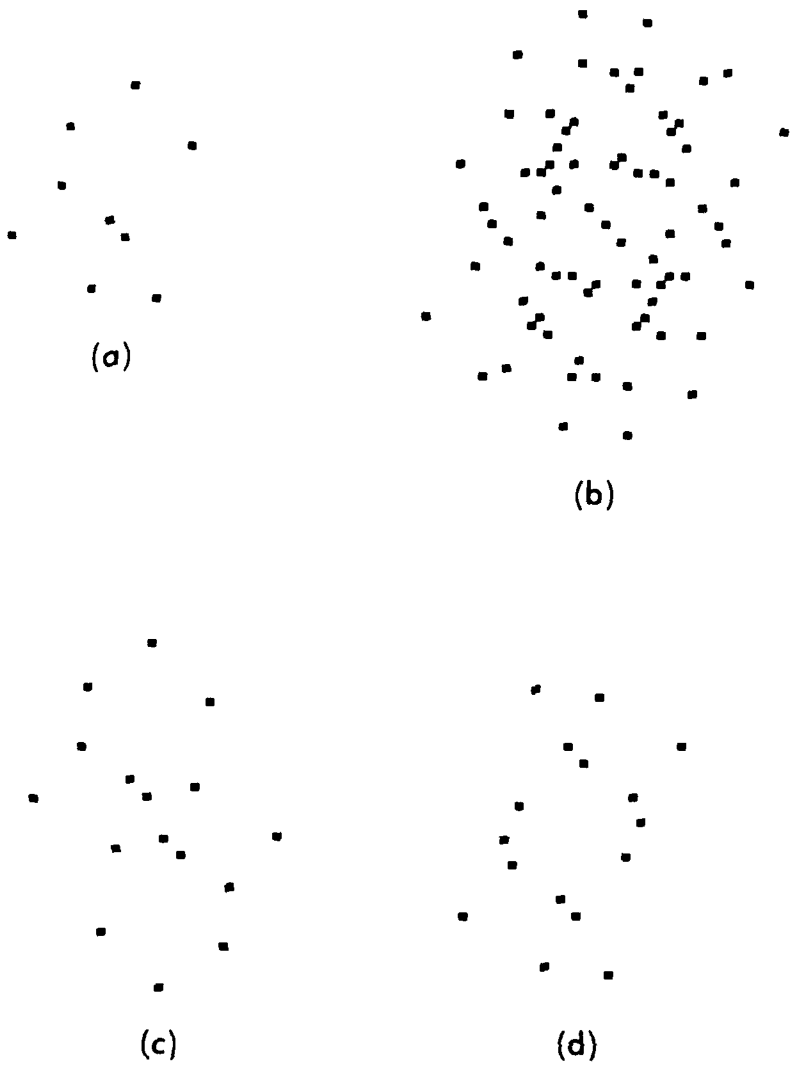

Figure 6. Intersection of Sets Consisting of a Collection of Points. (a) Set S; (b) $A = S-S$; (c) and (d) locators of the form $L = A \cap (w + A)$.

It can also be shown that even when Condition 1 is not satisfied it is sometimes possible to find solutions (and the solutions may even be unique as it was in Example 5) by intersecting A with itself three or more times. However, when Condition 1 is not satisfied, then there is no guarantee that the solution is unique, and finding solutions is considerably more complicated than simply evaluating B by Eq. (29). Unfortunately, given A it is not possible to immediately determine whether Condition 1 is satisfied. A necessary condition that Condition 1 (or Condition 3) be satisfied is that the number of points in A can be expressed as $N^2 - N + 1$ where $N \geq 1$ is an integer.

3.9 RECONSTRUCTION OF POINT-LIKE OBJECTS

By a simple modification of the method described in the previous section for reconstructing the support of a point-like object, it is often possible to reconstruct the object itself. The method is analogous to using Eq. (29) to compute B, except that it deals with products of autocorrelation functions instead of intersections of autocorrelation supports.

Suppose that the object is given by Eq. (27), consisting of N delta functions located at the distinct points x_n having amplitudes f_n, $n = 1, 2, \ldots, N$. The positions x_n are vectors in any number of dimensions. The autocorrelation is

$$\begin{aligned} f\bigstar f(x) &= \int_{-\infty}^{\infty} f(y)\, f(y+x)\, dV(y) \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} f_n f_m \delta(x - x_m + x_n) \end{aligned} \tag{32}$$

which can be expressed as

$$f\bigstar f(x) = \left(\sum_{n=1}^{N} f_n^2 \right) \delta(x) + \sum_{n=1}^{N} \sum_{m \neq n} f_n f_m \delta(x - x_m + x_n) \tag{33}$$

which has N^2 terms located at positions $x = x_m - x_n$, N of which are at $x = 0$. That is, it has up to $N^2 - N + 1$ distinct terms.

At this point we would like to take the product of two such autocorrelation functions; however, the product of two delta functions is not well defined. In order to overcome this problem, we define the product of two delta functions as follows:

$$[a\delta(x - x_1)][b\delta(x - x_2)] \equiv \begin{cases} ab\delta(x - x_1), & x_2 = x_1 \\ 0 & , x_2 \neq x_1 \end{cases}$$

Now consider multiplying $f\bigstar f(x)$ by $f\bigstar f(x - x_1 + x_k)$, where $x_1 - x_k \neq 0$ lies within the support of $f\bigstar f(x)$. The center of the translated autocorrelation lies within the support of the untranslated autocorrelation. This gives the autocorrelation product (all summations are from 1 to N unless otherwise noted)

$$AP_{1k}(x) = [f \bigstar f(x)][f \bigstar f(x - x_1 + x_k)]$$

$$= \left[\left(\sum_n f_n^2\right)\delta(x) + \sum_n \sum_{m\neq n} f_n f_m \delta(x - x_m + x_n)\right] \tag{34a}$$

$$\cdot\left[\left(\sum_n f_n^2\right)\delta(x - x_1 + x_k) + \sum_{n'} \sum_{m'\neq n'} f_{n'} f_{m'} \delta(x - x_{m'} + x_{n'} - x_1 + x_k)\right] \tag{34b}$$

$$= \left(\sum_n f_n^2\right) f_1 f_k \delta(x) + \left(\sum_n f_n^2\right) f_1 f_k \delta(x - x_1 + x_k)$$

$$+ f_1 f_k \sum_{m\neq k,1} f_m^2 \delta(x - x_m - x_k)$$

$$+ f_1 f_k \sum_{n\neq k,1} f_n^2 \delta(x - x_1 + x_n) + (\text{O.T.})$$

where (O.T.) denotes "other terms" as will be described later. [As an example of how Eq. 34(b) follows from Eq. 34(a), the fourth term in Eq. 34(b) arises from the product of the second term of the first autocorrelation with the second term of the second autocorrelation, with m = 1, n' = n, and m' = k.] From another way of expressing Eq. (34)

$$AP_{1k}(x) = \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{n'=1}^{N} \sum_{m'=1}^{N} f_n f_m f_{n'} f_{m'} \delta(x - x_m + x_n)$$

$$\cdot \delta(x - x_{m'} + x_{n'} - x_1 + x_k) \tag{34c}$$

it is seen that terms survive at points

$$x = x_m - x_n = x_{m'} - x_{n'} + x_1 - x_k \tag{35}$$

The terms shown in Eq. 34(b) all necessarily appear. In addition, other terms may appear, as indicated by "+ (O.T.)". The existence other terms depends on the presence of special relationships between the coordinates x_n allowing Eq. (35) to be satisfied. There being no additional terms is equivalent to Condition 1 (described in the previous section) being satisfied. If the x_n were independent random variables, then the chance of having additional surviving terms would be small, and we would have (O.T.) = 0.

Combining Eq. (27) with 34(b), the autocorrelation product can be expressed as

$$AP_{1k}(x) = f_1 f_k \left[f^2(x + x_k) + f^2(-x + x_1)\right]$$

$$+\left(\sum_{n\neq k,1} f_n^2\right) f_1 f_k [\delta(x) + \delta(x - x_1 + x_k)] + (O.T.) \tag{36}$$

Therefore, translates of the supports of both f(x) and f(-x) are contained within the support of $AP_{1k}(x)$. This can also be seen from the fact that by Eq. (16) the support of $AP_{1k}(x)$ is a locator set.

Included in the support of $AP_{1k}(x)$ are points $x_n - x_k$ and $x_1 - x_n$, $n = 1, 2, \ldots, N$. Therefore the center of $f{\bigstar}f(x - x_{1'} + x_k)$, $1' \neq 1, k$ is within the support of $AP_{1k}(x)$. If $(U.T.) = 0$, then the product of the three autocorrelations is

$$AP_{1k1'k}(x) = [f{\bigstar}f(x)].[f{\bigstar}f(x - x_1 + x_k)].[f{\bigstar}f(x - x_{1'} + x_k)]$$

$$= AP_{1k}(x)\,[f{\bigstar}f(x - x_{1'} + x_k)]$$

$$= f_1 f_k \left[\left(\sum_n f_n^2\right)\delta(x) + \left(\sum_n f_n^2\right)\delta(x - x_1 + x_k) + \sum_{n\neq k,1} f_n^2\delta(x - x_n + x_k) + \sum_{n\neq k,1} f_n^2\delta(x - x_1 + x_n)\right]$$

$$\cdot\left[\left(\sum_n f_n^2\right)\delta(x - x_{1'} + x_k) + \sum_{n'}\sum_{m'\neq n'} f_{n'}f_{m'}\,\delta(x - x_{m'} + x_{n'} - x_{1'} + x_k)\right]$$

$$+ f_k f_1 f_{1'}\left\{\sum_{n\neq k,1,1'} f_n^3\delta(x - x_n + x_k) + \left(\sum_n f_n^2\right)[f_k\delta(x) + f_1\delta(x - x_1 + x_k) + f_{1'}\delta(x - x_{1'} + x_k)]\right\} \qquad (37a)$$

$$= f_k f_1 f_{1'}\left[f^3(x + x_k) + \left(\sum_{n\neq k} f_n^2\right)f_k\delta(x) + \left(\sum_{n\neq 1} f_n^2\right)f_1\delta(x - x_1 + x_k) + \left(\sum_{n\neq 1'} f_n^2\right)f_1\delta(x - x_{1'} + x_k)\right] \qquad (37b)$$

That is, the support of the product of three autocorrelations has the same support as $f(x + x_k)$, as was shown earlier in connection with Eq. (29), since B is just the support of the product of three

such autocorrelation functions. Furthermore, except at three points the product is proportional to the cube of $f(x + x_k)$.

The values at all points can be determined as follows: First,

$$\sum_n f_n^2 = f\bigstar f(0) \quad D \tag{38}$$

is known, so that factor can be divided out from the last three terms of Eq. 37(a). Second, let the coefficients of those three terms in Eq. 37(a) be (with $\sum f_n^2$ divided out)

$$A = D^{-1}\, AP_{1kl'k}(0) = f_k^2\, f_l\, f_{l'} \tag{39a}$$

$$B = D^{-1}\, AP_{1kl'k}(x_l - x_k) = f_k\, f_l^2\, f_{l'} \tag{39b}$$

$$C = D^{-1}\, AP_{1kl'k}(x_{l'} - x_k) = f_k\, f_l\, f_{l'}^2 \tag{39c}$$

Solving, we get

$$f_k = \left(\frac{A^3}{BC}\right)^{1/4} \tag{40a}$$

$$f_l = \left(\frac{B^3}{AC}\right)^{1/4} \tag{40b}$$

$$f_{l'} = \left(\frac{C^3}{AB}\right)^{1/4} \tag{40c}$$

and

$$f_k f_l f_{l'} = (ABC)^{1/4} \tag{40d}$$

The remaining f_n's, for $n \neq k, l, l'$ can then be computed by dividing Eq. 37(a) by $f_k f_l f_{l'}$ and then taking the cube root:

$$f_n = \left[\frac{AP_{1kl'k}(x_n - x_k)}{f_k\, f_l\, f_{l'}}\right]^{1/3} \tag{40e}$$

By this method f(x) is reconstructed exactly to within a translation, as long as (O.T.) = 0.

In performing these calculations, had we chosen a translation of the form $(x_l - x_{k'})$, $k' \neq k,l$ instead of $(x_{l'} - x_k)$, then the result would have been similar, except a translate of f(-x) would have been reconstructed instead of a translate of f(x). If (O.T.) $\neq$ 0, that is, if Condition 1 is not satisfied, then additional terms appear that make the analysis much more complicated and may prevent the reconstruction of f(x).

Various modifications to this reconstruction method are possible. For example, the product of two autocorrelation products $AP_{lk}(x)$ $AP_{l'k}(x)$ is proportional to $f^4(x + x_k)$ except at three points. Another example is to define the autocorrelation support function as

$$A(x) = \delta(x) + \sum_{n=1}^{N} \sum_{m \neq n} \delta(x - x_m + x_n) \tag{41}$$

which is just a binary-valued version of Eq. (33). Then the product of the autocorrelation function with two properly translated autocorrelation support functions is proportional to a translate of f(x), except at a single point which can be determined by a few extra simple steps.

In arriving at Eq. (37), it was assumed that the other terms (O.T.) = 0, or equivalently that Condition 1 be satisfied. The terms included in Eqs. 34(b) and (37) are those that necessarily arise by satisfying

$$x_m - x_n = x_{m'} - x_{n'} + x_l - x_k \tag{42}$$

trivially, for example, for m = n, m' = k, and n' = l. The other terms are those that satisfy Eq. (42) by chance, that is, those that arise in addition to those that (trivially) arise necessarily. These other terms require a special relationship between the points in S, and would not be expected to occur if the points in S are randomly distributed in some region of E^N.

These results, with some modifications, can also be extended to the case of an object having support on a number of disjoint islands having diameters small compared with their separations (as opposed to the support consisting of isolated mathematical points). However, as the number of islands increases and as the ratio of the diameters of the islands to their separations increases, the probability of satisfying a condition analogous to Condition 1 decreases.

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APPENDIX A

ASTRONOMICAL IMAGING BY PROCESSING STELLAR SPECKLE INTERFEROMETRY DATA

by

J.R. Fienup and G.B. Feldkamp

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Astronomical imaging by processing stellar speckle interferometry data

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Abstract

Diffraction-limited images, of resolution many times finer than what is ordinarily obtainable through large earth-bound telescopes, can be obtained by first measuring the modulus of the Fourier transform of an object by the method of Labeyrie's stellar speckle interferometry, and then reconstructing the object by an iterative method. Before reconstruction is performed, it is first necessary to compensate for weighting functions and noise in order to arrive at a good estimate of the object's Fourier modulus. A simple alternative to Labeyrie's method of compensation for the MTF of the speckle process is described. Experimental reconstruction results are shown for the binary star system SAO [illegible].

Introduction

As discussed in several papers in this session on Stellar Speckle Interferometry, Labeyrie's[1] method of processing many short-exposure images can be used to arrive at a diffraction-limited estimate of the Fourier modulus of an astronomical object despite the presence of atmospheric turbulence. Since the diffraction limit of a large-aperture telescope is many times finer than the resolution ordinarily obtainable through the atmosphere at optical wavelengths[2], stellar speckle interferometry has the potential for providing images having many times finer detail than what is ordinarily obtainable from earth-bound telescopes. Unfortunately, except for special cases in which an unresolved star is very near the object of interest, the Fourier modulus data can be used to directly compute only the autocorrelation of the object and not the object itself. In recent years, it has been shown that this stumbling block can be overcome by an iterative method[3] of computing the object's spatial (or angular) brightness distribution, which uses the Fourier modulus data provided by stellar speckle interferometry combined with the a priori knowledge that the object's distribution is nonnegative. This method provides an alternative to other fine-resolution imaging techniques[4,5].

In the remainder of this paper, stellar speckle inteferometry and the iterative reconstruction method are briefly reviewed. Then more detailed discussions of noise terms and MTFs that are inherent in speckle interferometry are given, and methods of obtaining an improved estimate of an astronomical object's Fourier modulus are described. Finally, some reconstruction results obtained with telescope data are shown.

Basic stellar speckle interferometry

Labeyrie's[1] stellar speckle interferometry starts by taking a number of short-exposure images of an astronomical object:

$$d_m(x) = f(x) * s_m(x) \tag{1}$$

where $f(x)$ is the spatial or angular brightness distribution of the object and $s_m(x)$ is the point-spread function due to the combined effects of atmosphere and the telescope for the [illegible] exposure. The coordinate x is a two-dimensional vector and $*$ denotes convolution. It is assumed that the exposure time is short enough to "freeze" the atmosphere and that a narrow spectral band is used. The Fourier transform of each short-exposure image is computed:

$$D_m(u) = \int_{-\infty}^{\infty} d_m(x) \exp(i2\pi ux)\, dx \tag{2}$$

In this paper, capital letters will denote the complex Fourier transforms of the corresponding lower-case letters, and the coordinate u is referred to as a spatial frequency. The summed squared Fourier modulus (the summed power spectrum) is computed:

$$\sum_{m=1}^{M} |D_m(u)|^2 = \sum_{m=1}^{M} |F(u) S_m(u)|^2 = |F(u)|^2 \sum_{m=1}^{M} |S_m(u)|^2 \tag{3}$$

[illegible] $\sum$ [illegible] can be thought of as the square of the MTF of the speckle interfer- [illegible] (the speckle MTF²) and it can be determined approximately by performing [illegible] speckle interferometry on an isolated unresolved star through atmospheric condi- [illegible] the same statistics as those through which the object imagery is taken. [illegible] power spectrum by the speckle MTF² yields, according to Eq. (3), [illegible] Fourier modulus of the object.

[illegible] the Fourier transform of $|F(u)|^2$, the autocorrelation of the object [illegible] method. However, the autocorrelation gives only very limited [illegible] its diameter, and the separation for the case of a binary [illegible] system. [illegible], for the special cases of (1) an object known to be centro-symmetric and [illegible] object having an isolated unresolved star within the same isoplanatic patch² [illegible] can the autocorrelation, or equivalently $|F(u)|^2$, be used [illegible] compute the object.

The iterative method

We have, in addition to the measured Fourier modulus, the a priori knowledge that the [illegible] real, nonnegative function. The reconstruction problem consists of [illegible] nonnegative object that is consistent with the measured Fourier modulus data. [illegible] problem can be solved by the iterative method depicted in Figure 1. It consists of

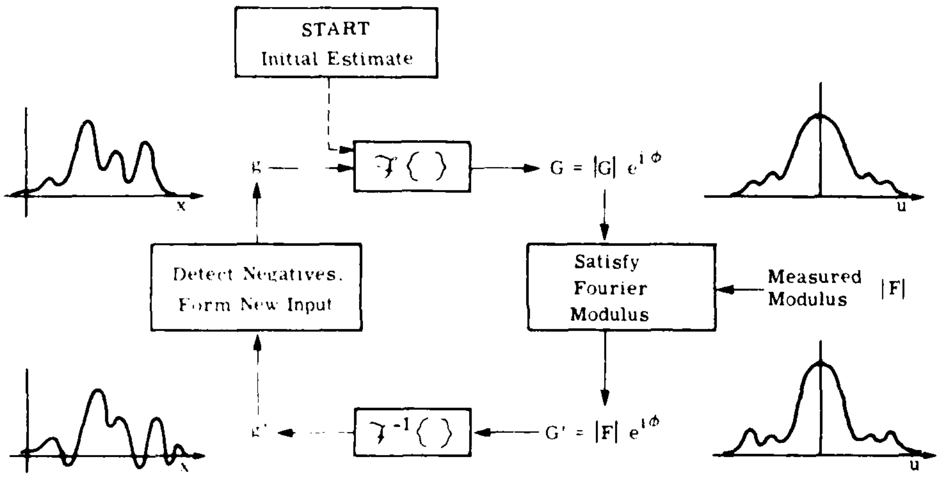

Figure 1. Iterative processing overview.

[illegible] object, $g(x)$ (which we usually choose to be a [illegible] transformed; (2) in the Fourier domain, the measured [illegible] computed Fourier modulus, and the computed phase is [illegible] Fourier transformed, yielding an image $g'(x)$; and (4) [illegible] the object-domain constraints by $g'(x)$. [illegible] mean-squared error is reduced to a small value con- [illegible] with the signal-to-noise ratio of the measured Fourier modulus data. The mean- [illegible] error in the image domain is

$$E_0^2 = \frac{\int_\gamma [g'(x)]^2\, dx}{\int_{-\infty}^{\infty} [g'(x)]^2\, dx} \tag{4}$$

where the region γ includes all points at which $g'(x)$ violates the object domain con- straints (where it is negative or possibly where it exceeds an a priori known diameter). Several different methods for choosing a new $g(x)$ have proven successful. For the results shown in this paper, we used for most iterations

$$g_{k+1}(x) = \begin{cases} g_k'(x) & , \quad x \notin \gamma \\ g_k(x) - .5g_k'(x), & x \in \gamma \end{cases} \tag{5}$$

[illegible] with

$$g_{k+1}(x) = \begin{cases} g'_k(x), & x \notin \gamma \\ 0, & x \in \gamma \end{cases} \qquad (6)$$

[illegible] few iterations, where the subscript k refers to the k^{th} iteration and γ is [illegible] [illegible]. More detailed discussions of the iterative method and why it works can be [illegible] references [illegible].

[illegible] results shown later, when random numbers were used for the initial input, then over a hundred iterations were required for an array size of 128 × 128 pixels [illegible] the same number of iterations that had been required for complicated two-dimensional objects), taking about two minutes on a Floating Point Systems AP 120B array processor. When a [illegible] star pattern with the correct spacing (which can be determined from the autocorrelation) but the incorrect brightness ratio was used as the initial input, then only a [illegible] iterations were required for convergence, taking about 10 seconds. It was found that the nonnegativity constraint was sufficient, and the diameter constraint was not needed.

Noise and MTF characteristics and their compensation

The data used for the experiments described here were obtained from the Steward Observatory [illegible] Interferometry Program which is described in more detail elsewhere in this [illegible] volume. For this "event detection" data, it is assumed that any one short exposure image contains no more than one photon in any one pixel (and most pixels record [illegible]). After an image is magnified, intensified, and detected, (among other things) it is thresholded to produce an image consisting of ones (where above the threshold) and zeros. Each image is autocorrelated, and the sum of all the autocorrelations is computed. The summed power spectrum is computed as the Fourier transform of the summed autocorrelation. In addition, each image is centroided (translated to make their centroids [illegible]) to within the nearest pixel, and the sum of the centroided images is computed.

The telescope diameter is 2.3 meters; and, for 30X magnification of the image, the image [illegible] approximately [illegible] arc-sec per pixel along each line, and is about 0.017 arc-sec [illegible] about 13% in that dimension relative to the along-line dimension. For [illegible] magnification, the scale is half that. The data is digitized in 256 × 256 arrays.

Figure [illegible] shows an example of a cut through the summed power spectrum of an unresolved star, [illegible]. This data was taken at 60X magnification (0.01 arc-sec per pixel = 4.7 × 10^{-8} radians per pixel) at a [illegible] nm wavelength band centered at 750 nm. The scale in the [illegible] is [illegible] $nm/(4.7 \times 10^{-8}$ rad × 256) = 0.062 meters (of telescope aperture) per pixel. For a telescope diameter of 2.3 meters, the highest spatial frequency passed by the telescope aperture is $2.3\ m/(0.062\ m/pix)$ = 37 pixels from zero frequency. Ideally (with [illegible] aberrations and no noise), the summed power spectrum of an unresolved star would be the [illegible] of the MTF due to the telescope aperture (that MTF is the autocorrelation of the telescope pupil function). Assuming a circular aperture, a cut through the telescope aperture MTF would have a roughly cone shape[8] and be zero beyond pixel 37. However, the summed power spectrum of the unresolved star shown in Figure 2 is very far from this ideal.

Two effects dominate the shape of the power spectrum. First, the speckle MTF^2, mentioned earlier in connection with Eq. (3), drops very rapidly for the very low spatial frequencies near the atmospheric cut-off. This results in the spike-like behavior of the summed power spectrum for very low spatial frequencies. Beyond the very low spatial-frequency region, the speckle MTF^2 is much better behaved and decreases slowly[5]. Second, photon noise results in, among other things, a noise bias term in the summed power spectrum[1]. This noise bias term dominates in the higher spatial frequencies. Beyond a radius of 37 pixels in the summed power spectrum, no signal energy exists -- it is purely noise. They so dominate the summed power spectrum that little useful information can be obtained unless compensation is made for both of these two effects.

The noise bias term and the detection transfer function

One would ordinarily eliminate the noise bias term simply by subtracting a constant from the summed power spectrum[10,11]. However, as seen from Figure 2, the noise bias term, which is seen by itself beyond pixel 37, is not a constant in this case. This results from the fact that upon detection and thresholding, a single photon sometimes results in more than one pixel recording a one, depending upon the threshold level and the size of the [illegible] of light exiting from the image intensifier. Table 1 shows the autocorrelations and the individual squared transfer functions of some of the various patterns of ones resulting from a single photon. Each pattern is, in effect, the impulse response of the de-

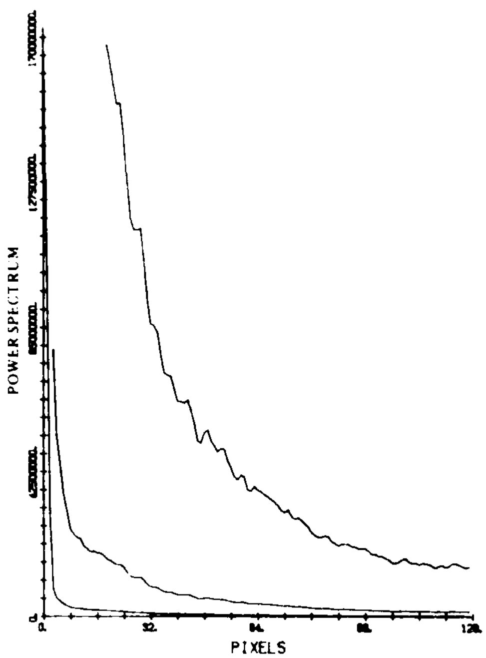

Figure 2. Summed power spectrum of an unresolved star (linear scale). The middle and upper curves are the same as the lower curve, except have 10X and 100X vertical scales, respectively.

Table 1. Event detection data: individual impulse responses, their auto correlations, and their power spectra.

DETECTION IMPULSE RESPONSE	AC OF IMPULSE RESPONSE	DETECTION TRANSFER FUNCTION²
1	1	1
1 1	1 2 1	$2 + 2\cos(2\pi\, u/N)$
1 / 1 (vertical)	1 / 2 / 1 (vertical)	$2 + 2\cos(2\pi\, v/N)$
1 1 1	1 2 3 2 1	$3 + 4\cos(2\pi\, u/N) + 2\cos(4\pi\, u/N)$
1 / 1 1	1 1 / 1 3 1 / 1 1	$3 + 2\cos(2\pi\, u/N) + 2\cos(2\pi\, v/N) + 2\cos\left[2\pi(u+v)/N\right]$
⋮	⋮	⋮

tection system; and in any one image, several different patterns may appear. That is, this impulse response may vary from photon to photon within the same image. Assuming a sparse population of photons within each image, it can be shown that the net squared transfer function, due to the ensemble of photon-produced patterns within an image, is given by a weighted sum of the individual squared transfer functions of the individual patterns. We refer to this weighted sum as the detection transfer function squared (DTF^2).

One can compensate for the noise bias term by the following steps[12]. (1) Over the spatial frequencies above the telescope cut-off, perform a two-dimensional least-squares fit of a weighted sum of individual squared transfer functions (some of which are shown in Table 1) to the summed power spectrum. By this, the DTF^2 is determined. (2) Compensate the effect of the DTF^2 by dividing the summed power spectrum by the DTF^2 (for all spatial frequencies). By this, the noise bias term is made a constant. (3) Subtract from the DTF^2-compensated summed power spectrum the constant noise bias term. This DTF^2 and noise bias compensation are demonstrated in Figures 3 and 4 for the binary star system ADS 14163. In this case, the magnification was 30X and the wavelength was 750 nm (10 nm spectral bandwidth) and so the telescope cut-off is at a spatial frequency of 74 pixels. This data set resulted from power-spectrum averaging of 1820 short exposure images containing a total of about 2.4×10^5 photons.

In the autocorrelation domain, the noise bias term results in a spike at the (0, 0) coordinate, and the DTF^2 causes the spike to be spread over a few pixels about (0, 0). Compensation for the DTF^2 causes the spike to collapse to a delta-function at (0, 0). Then the subtraction of the noise bias in the Fourier domain removes the delta-function at (0, 0) in the autocorrelation.

More generally, the functional form of the DTF^2 is heavily dependent on the manner in which the images are detected and should be modified according to the characteristics of the detection hardware used.

The speckle MTF^2

Compenstion for the speckle MTF^2 would ordinarily be accomplished by dividing the summed power spectrum by the summed power spectrum of a reference star[1]. Both power spectra should first be corrected for the DTF^2 and the noise bias term.

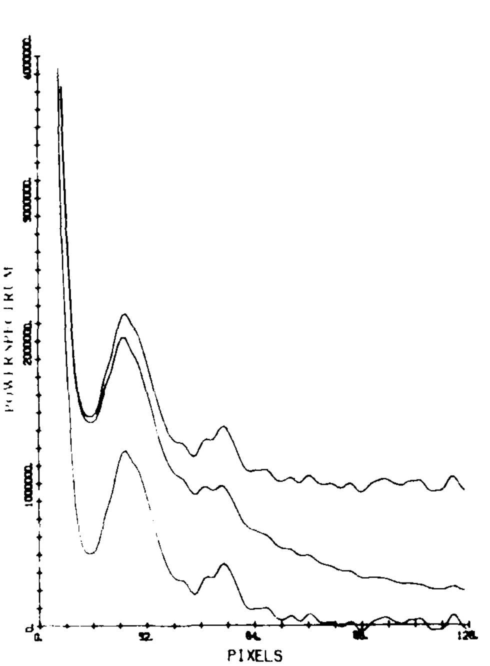

Figure 3. Power spectrum of the binary SAO [illegible]. (A) Middle curve: raw summed power spectrum; (B) upper curve: summed power spectrum divided by the DTF^2; (C) lower curve: DTF^2 - compensated summed power spectrum with noise bias subtracted.

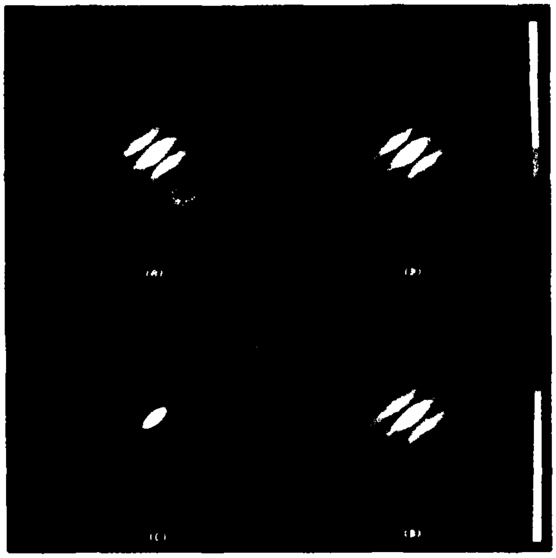

Figure 4. (A)-(C) Two dimensional view of Figure 3(A)-(C), (D) the Fourier modulus, i.e., the square root of (C). Note: the residual *noise beyond the telescope cut-off frequency* is visible in this case and not in (C) because the square root operation reduces the dynamic range of the *data*.

[illegible], there is not available the summed power spectrum of a reference star [illegible] time to the observation [illegible] actual speckle MTF [illegible] fitting [illegible] alternative is the Worden subtract method [illegible]. [illegible] effectiveness of this method[14]; [illegible] estimate of the object's power spectrum than [illegible] to determine whether it [illegible].

[illegible] subtract method of compensating for the speckle MTF [illegible] summed autocorrelation the sum of cross-correlations of different short-exposure images. The short-exposure images must be centroided [illegible]. [illegible] shown that subtraction of a properly scaled version of the power spectrum of [illegible] centroided images from the summed power spectrum [illegible] the Worden subtract method. Figure 5(a) shows a cut through these two [illegible] very low spatial frequencies for the unresolved star [illegible] like behavior for very low spatial frequencies. However, [illegible] from zero frequency and beyond, the power spectrum [illegible] is essentially zero. Thus, it would appear that the Worden subtract method [illegible] for the very lowest spatial frequencies. [illegible] recalling that the compensated summed power spectrum for [illegible] star should be a constant (weighted by the MTF [illegible] nearly unity for these very low spatial frequencies), we see from Figures [illegible] Worden subtract method did not produce the expected result. [illegible] the low-frequency spike in the summed power spectrum, the Worden subtract method [illegible] its mean-squared error; however, it did not replace the spike with [illegible] low-frequency information. It is not presently known whether this [illegible] of the Worden subtract method is due to [illegible] with this particular data set; however, it is consistent with the analysis [illegible].

A better speckle MTF compensation than the Worden subtract method [illegible] simply be to clip the summed power spectrum at the very low spatial frequencies. That is, [illegible] that the object's power spectrum is nearly constant for the very low spatial fre-

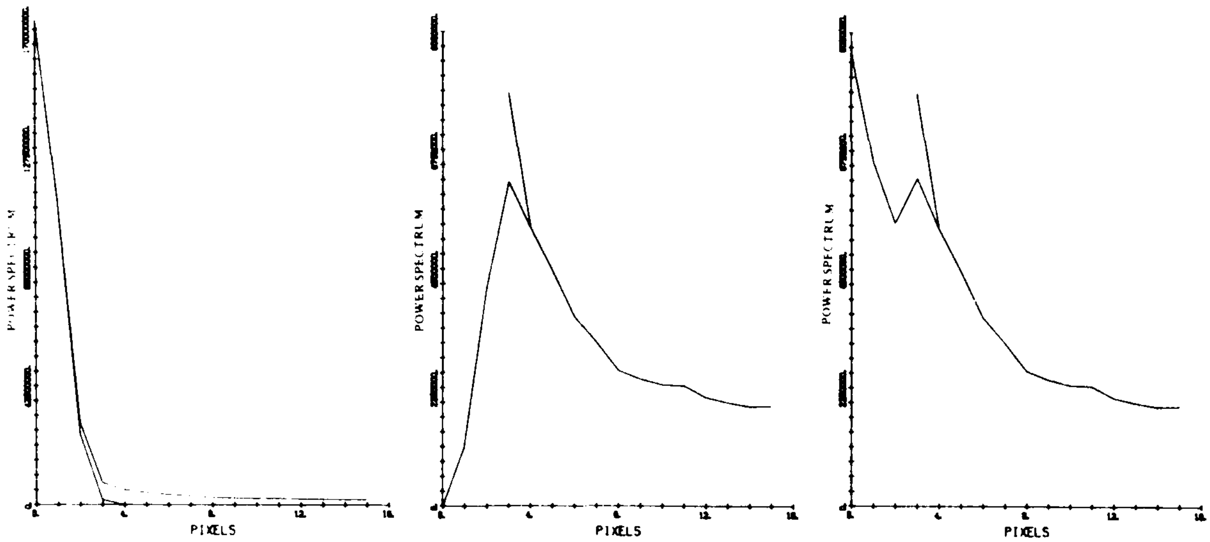

Figure 5. Worden subtract method on an unresolved star (low spatial frequencies). (A) upper curve: summed power spectrum; lower curve: power spectrum of the sum of the centroided images; (B) upper curve: summed power spectrum (note expanded vertical scale); lower curve: summed power spectrum minus the power spectrum of the sum of the centroided images; (C) same as (B), except a smaller percentage of the power spectrum of the sum of the centroided images was subtracted.

quencies (which would be true for objects of diameter only a small fraction of an arc-sec), we replace the summed power spectrum in that region by a constant. The constant is chosen to be consistent with the value of the summed power spectrum in the region just beyond the very low-frequency spike. As in the case of the Worden subtract method, this method does not correct for the middle-frequency vs. higher-frequency regions of the speckle MTF^2; however, as noted earlier, the speckle MTF^2 is reasonably well behaved for those spatial frequencies, and correcting for the very low spatial frequencies corrects for the greatest part of the error.

The method of clipping the summed power spectrum to correct for the speckle MTF^2 is demonstrated in Figure 6 for the binary SAO 94163 for which reference star data was not available. In order to increase the accuracy of the assumption that the Fourier modulus (or its square, the power spectrum) is constant for very low spatial frequencies, the OTF and noise-bias-corrected Fourier modulus was divided by the MTF due to the telescope aperture (which was approximated by the MTF due to a circular aperture of diameter 2.3 meters). The elliptical shape of the Fourier modulus data is due to the difference in scale factors in the two dimensions as noted earlier. Within the low frequency region, wherever the Fourier modulus exceeded a threshold value, it was clipped to that threshold value. The result was multiplied by the MTF due to the telescope to arrive at our final estimate of the Fourier modulus of SAO 94163 including the telescope MTF. In the process of multiplying back in the telescope MTF, the residual noise beyond the telescope cut-off frequency was set to zero.

Image reconstruction results

The Fourier modulus estimate shown in Figure 6(d) was truncated to a 128 x 128 array, in order to save computation time in the iterative reconstruction. This caused a slight truncation of the highest spatial frequencies along the horizontal dimension of Figure 6(d). SAO 94163 was reconstructed using the iterative method, and the images resulting from two different selections of the initial input to the algorithm are shown in Figures 7(a) and (b), respectively. The rms error E_0 was reduced to about 0.05. For the purpose of display, a (sin x)/x interpolation was performed on the images of Figure 7 in order to increase the sampling rate across the image. In order to get an indication of the sensitivity of the method to the clipping threshold level described in the previous section, the clipping was done over again using a 33% greater threshold value (which is obviously greater than the optimum threshold). Two images reconstructed from the resulting Fourier modulus estimate are shown in Figure 7(c) and (d). Half the time, the iterative reconstruction algorithm produces an image rotated by 180° due to the inherent 2-fold ambiguity of the Fourier modulus data.

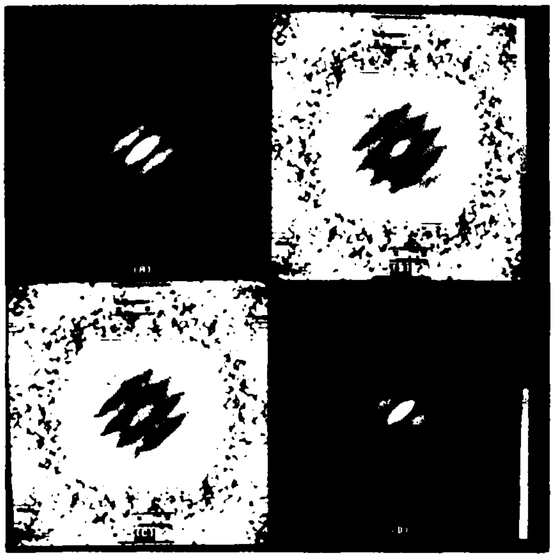

Figure 6. Clipping to compensate for the speckle MTF (for "seeing") for the binary SAO 94163. (A) Fourier modulus, same as Figure 4(D), (B) Fourier modulus compensated for telescope MTF (attempted division by zero is evident for spatial frequencies above the telescope cut-off frequency); (C) clipping of the low spatial frequencies; (D) Fourier modulus estimate obtained by putting back in the telescope MTF.

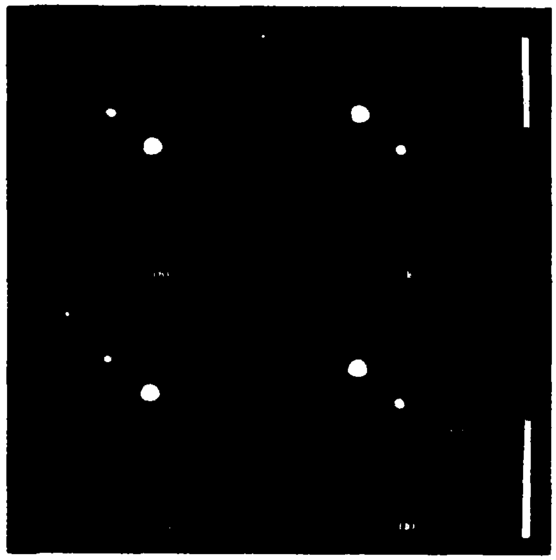

Figure 7. Reconstructed images of SAO 94163 (see text).

The average separation of the reconstructed images [illegible] arc-sec and the orientation angle is 42.7° (47.3° from the orthogonal axis). The brightness ratios, [illegible] the maximum brightness of each star in the pair, are [illegible], [illegible], 4.14, and [illegible] for Figures 7(a) through (d), respectively, and the corresponding magnitude differences are [illegible], [illegible], 1.[illegible], and 1.4[illegible], respectively. Thus, a [illegible] increase in threshold level [illegible] only a 6% decrease in the computed brightness ratio. The average magnitude difference of the reconstructions of Figures 7(a) and (c) is 1.57.

Conclusions

We have discussed steps necessary to obtain an accurate estimate of an object's Fourier modulus from the raw summed power spectrum: detection transfer function compensation, noise bias subtraction, and speckle MTF compensation. For compensation of the speckle MTF when reference star data is not available, an improvement over the wander subtract method is the simple method of clipping the Fourier modulus spike at the very low spatial frequencies (for objects much smaller than one arc-sec in diameter). A reconstruction of the binary SAO 94163 using this method resulted in an image having a binary separation of 0.27 arc-sec at an angle of 42.7° (47.3°) and a magnitude difference of 1.57 (brightness ratio of 4.26). The reconstruction of a binary is trivial and does not require the use of the iterative method; however, the iterative method is required for complicated objects, and the data processing steps described here for this simple example can be used in more general circumstances. Based on the success of image reconstruction experiments using summed power spectra of complicated two-dimensional objects computer-simulated to include the effects of atmospheric turbulence and photon noise[11], it is expected that it will be possible to reconstruct fine-resolution images of complicated astronomical objects.

Acknowledgments

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APPENDIX B

145400-4-J

PHASE RETRIEVAL FOR FUNCTIONS WITH DISCONNECTED SUPPORT

by

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and
J.R. Fienup

Abstract

The uniqueness of solutions to the phase retrieval problem is explored using the theory of analytic functions. It is shown that if a function or its autocorrelation satisfy certain disconnection conditions, then the solution is unambiguous unless the separated parts of the function are related to one another in a special way.

Physics and Astronomy Classification Scheme:	42.30. Kq
	42.30. Va
	02.30. +g
	42.10. Hc

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1. Introduction

The problem of phase retrieval arises in many fields: optical astronomy, radio astronomy, radar, antenna theory, holography, interferometry, crystallography, electron microscopy, Fourier transform spectroscopy, and shape recognition (Fourier descriptors). Given a solution to the phase retrieval problem, one would like to know if it is unique. The general one-dimensional phase retrieval uniqueness problem can be stated as follows: If f is a complex-valued function defined on the real line and F is its Fourier transform, under what conditions and to what extent is f determined by $|F|$? Since $|F|^2$ is the Fourier transform of the autocorrelation function of f, this problem can be equivalently stated in the form: Under what conditions and to what extent is f determined by its autocorrelation function?

In this paper, we restrict our attention to the case in which f is assumed to have compact support (i.e., $f(x) = 0$ for x outside of some finite interval). Methods of reducing the number of functions f having the same $|F|$ by imposing disconnectedness conditions on the support of f or the support of its autocorrelation function are explored. Some of the theorems presented in this paper are refinements and/or generalizations of known results and some are new. All proofs are in the appendix.

2. Terminology and Preliminaries

The _support_, S_f, of a complex-valued measurable function, f, on the real line, $\mathbb{R}$, is the smallest closed subset of the real line outside of which f is zero almost everywhere (a.e.). Since by changing its values on a set of measure zero a function can be made equal to zero everywhere off its support, in the remainder of this paper all functions will be assumed to be zero everywhere off their supports. The _interval of support_, $I(f) = [a_f, b_f]$, of f is the smallest closed interval containing S_f. The _center_, c_f, of f is given by $c_f = (a_f + b_f)/2$.

[illegible] subset of $\mathfrak{R}$ is compact if it is contained in some [illegible] interval. $\mathbb{C}$ is the complex plane. If $w \in \mathbb{C}$, then $\bar{w}$ is its complex conjugate. $L_0^2(\mathfrak{R})$ is the space of all complex-valued square-summable functions on $\mathfrak{R}$ with compact support.

If $f, g \in L_0^2(\mathfrak{R})$, the convolution of f and g is given by

$$f * g(x) = \int_{-\infty}^{\infty} f(y) g(x - y)\, dy.$$

The convolution of two functions in $L_0^2(\mathfrak{R})$ is also in $L_0^2(\mathfrak{R})$. We [illegible] $\tilde{f}(x) = \overline{f(-x)}$. The autocorrelation of f is given by

$$\text{auto}(f) = f * \tilde{f}.$$

The Laplace transform of f is given by

$$F(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx \quad \forall w \in \mathbb{C}.$$

Capital letters will be used to denote the Laplace transforms of the functions denoted by the corresponding lower case letters. If F is a function on $\mathbb{C}$, F^* is defined by $F^*(w) = \overline{F(\bar{w})}$. If F is the Laplace transform of f, then F^* is the Laplace transform of $\tilde{f}$. Also, if G is the Laplace transform of g, then the Laplace transform of $f * g$ is [illegible]. Therefore, the Laplace transform of $\text{auto}(f) = f * \tilde{f}$ [illegible] F^*.

[illegible] function on $\mathbb{C}$, the function n_F is defined [illegible] $\mathbb{C}$ [illegible] w:

1. If w is not a zero of F then $n_F(w) = 0$.
2. If w is a non-real zero of F then $n_F(w)$ is its order.
3. If w is a real zero of F then $n_F(w)$ is one-half its order.

[illegible] and $w \in \mathfrak{R}$ [illegible],

$$\mathcal{D}(F) = \left\{ w: \ \operatorname{Im}(w) \geq 0, \ \eta_{FF^*}(w) > 0 \text{ and } w \neq 0 \right\},$$

$$\tau_F(w) = \eta_F(w) - \eta_F(\overline{w}),$$

and

$$\mathcal{W}(F) = \left\{ w: \ \tau_F(w) > 0 \right\}.$$

If A and B are sets,

$$A \setminus B = \left\{ x: \ x \in A \text{ and } x \notin B \right\} .$$

If $A, B \subseteq \mathfrak{R}$, then

$$A - B = \left\{ a - b: \ a \in A \text{ and } b \in B \right\}$$

and

A' is the complement of A in $\mathfrak{R}$.

If a and c are real numbers, the functions $f(x)$, $f(x + a)e^{ic}$ and $\tilde{f}(x + a)e^{ic}$ all have the same Fourier modulus. If these are the only functions with that Fourier modulus, then $f(x)$ is said to be unique and its Fourier modulus is said to be unambiguous. Otherwise, f is non-unique and its Fourier modulus is ambiguous. If

$$g(x) = f(x + a)e^{ic} \text{ or } g(x) = \tilde{f}(x + a)e^{ic}$$

then f and g are equivalent or in symbols, $f \sim g$.

3. Examples of Non-Uniqueness

Let N be a positive integer and $A = \left\{1, 2, \ldots, N\right\}$. Let B be a subset of A and $C = A \setminus B$.

Theorem 1: Let b_n, $n = 1, \ldots, N$, and α be given complex numbers and let c_n and d_n, $n = 0, 1, \ldots, N$, be defined by

$$\alpha \prod_{n=1}^{N} (x - b_n) = \sum_{n=0}^{N} c_n x^n$$

and

$$\alpha\left[\prod_{n\epsilon B}(-\bar{b}_n x + 1)\right] \cdot \left[\prod_{n\epsilon C}(x - b_n)\right] = \sum_{n=0}^{N} d_n x^n$$

Let f be an arbitrary function in $L_0^2(\mathfrak{R})$, β a real number,

$$g(x) = \sum_{n=0}^{N} c_n f(x - n\beta)$$

and

$$h(x) = \sum_{n=0}^{N} d_n f(x - n\beta).$$

Then

$$|G(u)| = |H(u)| \ \forall\ u\epsilon\ \mathfrak{R}.$$

Note: A similar factorization technique is used by Bruck and Sodin in [illegible].

Example 1: Let

$$f(x) = \begin{cases} 1 \text{ for } |x| \leq 1/2 \\ 0 \text{ otherwise} \end{cases}$$

We have

$$(x + 3)(x + 2) = 6 + 5x + x^2$$

and

$$(3x + 1)(x + 2) = 2 + 7x + 3x^2.$$

Let

$$g(x) = 6f(x) + 5f(x - 4) + f(x - 8)$$

and

$$h(x) = 2f(x) + 7f(x - 4) + 3f(x - 8)$$

Then, by Theorem 1, $|G(u)| = |H(u)| \;\forall\; u \in \mathfrak{R}$ (see Figure 1).

In this example, the two functions are quite similar in appearance. In particular, they have the same support. In a particular application, it would probably be difficult to rule out either one as the right solution on the basis of a priori information.

<u>Example 2</u>: Let f be as in example 1. Now we use

$$2(x + 2)(x^2 - \frac{1}{2} x + 2) = 8 + 2x + 3x^2 + 2x^3$$

and

$$2(2x + 1)(x^2 - \frac{1}{2} x + 2) = 4 + 7x + 4x^3.$$

Let

$$g(x) = 8f(x) + 2f(x - 4) + 3f(x - 8) + 2f(x - 12)$$

and

$$h(x) = 4f(x) + 7f(x - 4) + 4f(x - 12)$$

(see Figure 2). Again, by Theorem 1, $|G(u)| = |H(u)| \;\forall\; u \in \mathfrak{R}$.

This example shows that, unlike the situation in Example 1, the functions can have different supports.

The following theorem gives another method for generating examples of non-uniqueness.

<u>Theorem 2</u>: Let $f, g \in L_0^2(\mathfrak{R})$ and

$$h_1 = f * g$$

$$h_2 = f * \tilde{g}.$$

Then

$$|H_1(u)| = |H_2(u)| \;\forall\; u \in \mathfrak{R}.$$

Examples 1 and 2 could have been obtained by using Theorem 2. In general, Theorem 2 is useful in generating examples of non-uniqueness in which both functions are positive because if f and g are positive, then so are h_1 and h_2. (Note: Theorem 2 also holds for functions of two or more variables.)

4. Examples of Uniqueness

Theorems 3 and 4 are presented in this section out of logical order because they are needed in the discussion of examples of unique functions. Their proofs (see appendix) use Corollary 3 to Theorem 8 [illegible].

By Corollary 3 to Theorem 3, a function f is unique if F has no non-real zeroes.

Example 3: Let f be as in Examples 1 and 2. Then

$$F(w) = 2 \operatorname{sinc} w = \begin{cases} 2 \dfrac{\sin w}{w} & \text{for } w \neq 0 \\ 2 & \text{for } w = 0. \end{cases}$$

Thus, F has no non-real zeroes and therefore f is unique.

The following theorem gives a method for generating more examples of uniqueness.

Theorem 3: Let $f \epsilon L_0^2(\mathfrak{R})$ be unique and let b_n be complex numbers with $|b_n| = 1$, $n = 1, \ldots, N$. Let c_n, $n = 0, 1, \ldots, N$, be defined by

$$\prod_{n=1}^{N} (x - b_n) = \sum_{n=0}^{N} c_n x^n$$

and let

$$g(x) = \sum_{n=0}^{N} c_n f(x - \beta n)$$

where β is a real number. Then g is unique.

Corollary: If f is unique and

$$g(x) = \sum_{n=0}^{N} f(x - \beta n)$$

then g is unique.

Example 4: Let f be as in the previous examples and let

$$g(x) = \sum_{n=0}^{N} f(x - 4n).$$

Then, by the corollary to Theorem 3, g is unique (see Figure 3).

Another method for generating examples of uniqueness is given by the following theorem.

Theorem 4: If f, $g \epsilon L_0^2(\mathfrak{R})$, f is unique and G has no non-real zeroes, then $f * g$ is unique.

Example 4: Let f be as in the previous examples and let $g = f * f$. Then, by Theorem 4, g is unique (see Figure 4).

5. Factorization of F

Let $f \epsilon L_0^2(\mathfrak{R})$. Then $\mathrm{auto}(f) = f * \tilde{f} \in L_0^2(\mathfrak{R})$ and it follows from the Paley-Weiner Theorem ([2], p. 103) that both F and FF^* are entire functions of exponential type. Since FF^* is entire, $D(F)$ (see Section 2) is countable. Let its elements be numbered

$$\Omega(F) = \left\{w_n : n = 1, 2, \ldots\right\}$$

such that $\left\{|w_n|\right\}_{n=1}^{\infty}$ is a non-decreasing sequence. The following factorization is essentially due to Titchmarsh ([3], p. 285, Theorem VI). (Note: In [3], Titchmarsh defines the Laplace transform as

$$F(w) = \int_{-\infty}^{\infty} f(x)\, e^{wx}\, dx$$

as opposed to our

$$F(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx.$$

Therefore, in using his results, appropriate adjustments must be made.)

Theorem 5: We have

$$F(w) = \frac{1}{m!} F^{(m)}(0) w^m e^{-ic_f w} \prod_{n=1}^{\infty} \left[\left(1 - \frac{w}{w_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(\bar{w}_n)} \right]$$

where $m = 2n_F(0)$ and the infinite product is conditionally convergent.

This particular form of factorization of F is chosen to facilitate "zero flipping" arguments which will be used later.

In the sequel, we will also need the following theorem.

Theorem 6: If β is an integer-valued function on $\mathbb{C} \ni 0 \le \beta(w) \le n_F(w)$ $\forall\ w \in \mathbb{C} \setminus \mathbb{R}$ and $0 \le \beta(u) \le 2n_F(u)$ $\forall\ u \in \mathbb{R}$, then the infinite product

$$\prod_{n=1}^{\infty}\left[\frac{1-\dfrac{w}{\bar{w}_n}}{1-\dfrac{w}{w_n}}\right]^{\beta(w_n)}$$

is absolutely convergent $\forall\ w\epsilon\ \mathfrak{C}$.

The next theorem has to do with "flipping zeroes." More will be said about this in the next section.

Theorem 7: Let $f\epsilon L_0^2(\mathfrak{R})$ and $\{w_n\}_{n=1}^{\infty}$ be the distinct non-real zeroes of F. Let β be an integer-valued function on $\mathfrak{C} \ni 0 \leq \beta(w) \leq n_F(w)$ for $w\epsilon\ \mathfrak{C} \setminus \mathfrak{R}$ and $0 \leq \beta(u) \leq 2n_F(u)$ for $u\epsilon\ \mathfrak{R}$. Let

$$G(w) = F(w)\prod_{n=1}^{\infty}\left[\frac{1-\dfrac{w}{\bar{w}_n}}{1-\dfrac{w}{w_n}}\right]^{\beta(w_n)}$$

and let g be the inverse transform of G. Then $|G(u)| = |F(u)|$ for $u\epsilon\ \mathfrak{R}$, $g\epsilon L_0^2(\mathfrak{R})$ and

$$I(g) = I(f).$$

6. Functions with Disconnected Support

In this section, it is shown that the probability of uniqueness is much higher for functions whose supports satisfy certain disconnectedness conditions.

Let I_n, $n = 1, \ldots, N$, be a sequence of disjoint closed intervals satisfying

6.1) $$(I_n - I_m) \cap (I_j - I_k) = \phi$$

for $j \neq k$ and $\langle n, m\rangle \neq \langle j, k\rangle$ (where $\langle\ ,\ \rangle$ denotes ordered pair). Let

$$A = \bigcup_{n=1}^{N} I_n.$$

Let $f,\ g \in L_0^2(\mathfrak{R})$ satisfy

$$S(f) \subseteq A \text{ and } S(g) \subseteq A$$

and let

$$f_n(x) = \begin{cases} f(x) \text{ for } x \in I_n \\ 0 \text{ otherwise} \end{cases}$$

and

$$g_n(x) = \begin{cases} g(x) \text{ for } x \in I_n \\ 0 \text{ otherwise} \end{cases}$$

for $n = 1, \ldots, N$. Finally, let

$$B = \bigcap_{n=1}^{N} Z(F_n)$$

we assume here that $f_n \not\equiv 0$, $n = 1, \ldots, N$.

Theorem 8: Let f, g, f_n, and g_n be as described above. $|F(u)| = |G(u)|\ \forall\ u \in \mathfrak{R}$ iff $\exists$ a real number θ and an integer-valued function α defined on $\mathfrak{C}$ with $0 \leq \alpha(z) \leq \min\limits_{1 \leq n \leq N} [n_{F_n}(z)]\ \forall\ z \in \mathfrak{C} \setminus \mathfrak{R}$ and $0 \leq \alpha(u) \leq 2 \min\limits_{1 \leq n \leq N} [n_{F_n}(u)]$ for $u \in \mathfrak{R}$ such that if

$$\varphi(w) = e^{i\theta} \prod_{z \in B} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\alpha(z)}$$

then for $N \neq 2$,

$$1)\quad G_n(w) = e^{i(c_f - c_g)w}\,\omega(w)F_n(w)$$

$$\text{and } c_{f_n} - c_{g_n} = c_f - c_g \text{ for } n = 1, \ldots, N,$$

and for $N = 2$, either 1) holds or

$$2)\quad G_n^*(w) = e^{i(c_f + c_g)w}\,\omega(w)F_n(w)$$

$$\text{and } c_{f_n} + c_{g_n} = c_f + c_g \text{ for } n = 1, 2.$$

If the function f is a more or less arbitrary function gotten from the real world (but satisfying the hypothesis of Theorem 8), then it should almost always be the case that $B = \phi$. For this case, we obtain the following result:

<u>Corollary 1</u>: If $B = \phi$, then $|F(u)| = |G(u)|$ $\forall$ $u \in \mathfrak{R}$ iff $f \sim g$.

In [4], Bates states a similar result but with too weak a hypothesis and too strong a conclusion. (See both the discussion immediately following and the discussion following Corollary 3 to Theorem 8.) It should be noted that Corollary 1 does not hold if it is merely assumed that the I_n are disjoint. To see this, let f, g and h be defined as in Example 1. Let

$$I_1 = [-0.5, 0.5],\ I_2 = [4.5, 5.5] \text{ and } I_3 = [7.5, 8.5].$$

Let g and h play the roles of f and g in Corollary 1. Then $Z(G_n) = \phi$ for $n = 1, 2, 3$ and hence $B = \phi$. Also, $|G(u)| = |H(u)|$ $\forall$ $u \in \mathfrak{R}$ but g and h are not equivalent.

By setting $N = 1$ in Theorem 8, we obtain the basic "zero-flipping" result of Hofstetter and Walther.

Corollary 2 (Hofstetter [5] - Waltner [6]): $F(u) = G(u)$ $\forall$ $u \in \mathbb{R}$ iff $\exists$ a real number θ and an integer-valued function α defined on $Z(F)$ with $0 \le \alpha(z) \le n_F(z)$ $\forall$ $z \in Z(F)$ such that

$$G(w) = e^{i\theta} e^{i(c_f - c_g)w} F(w) \prod_{z \in Z(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\alpha(z)}$$

Flipping all the non-real zeroes of F yields the following relations:

Lemma 1:

$$F^*(w) = \frac{\overline{F^{(m)}(0)}}{F^{(m)}(0)} e^{2ic_f w} F(w) \prod_{z \in Z(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{n_F(z)}$$

$$= \frac{\overline{F^{(m)}(0)}}{F^{(m)}(0)} e^{2ic_f w} F(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)}$$

where $m = 2n_F(0)$.

Since F^* is the Laplace transform of $\tilde{f}$ and $\tilde{f}$ is equivalent to f, we obtain:

Corollary 3: The following are equivalent:

1. $F(u)$ is unambiguous.
2. $F(w)$ has no non-real zeroes or one non-real zero of order 1.
3. $F(w)F^*(w)$ has no non-real zeroes or two non-real zeroes of order 1.

Corollary 1 suffers from the drawback that both f and g must be assumed to have their supports contained in $A = \bigcup_{n=1}^{N} I_n$. For this reason, it cannot be concluded that f is unique. The necessity of this condition can be seen from the following. Let f, g and h be as defined in Example 2. Let h play the role of the function f in Corollary 1. Also let

$$I_1 = [-0.5, 4.5] \text{ and } I_2 = [11.5, 12.5].$$

Then I_1 and I_2 satisfy the separation condition 6.1. Let h_n be the restriction of h to I_n, n = 1, 2. Then $I(h_2) = \emptyset$ and therefore $E = \emptyset$. Also $|H(u)| = |G(u)|$ $\forall\, u \in \mathbb{R}$ but h and g are not equivalent. Therefore, h is not unique. A similar result occurs in the N = 3 case if we let

$$I_1 = [-0.5, 0.5],\ I_2 = [3.5, 4.5] \text{ and } I_3 = [11.5, 12.5].$$

Another drawback is that the functions f and g of Corollary 1 are not known. Therefore the knowledge that their supports satisfy condition 6.1 must come in the form of a priori knowledge. The next theorem avoids these drawbacks for positive f by imposing a separation condition on the support of the autocorrelation function of f instead of the support of f. Since the autocorrelation function is known and is the same for any two functions with the same Fourier modulus, these drawbacks disappear.

First we need two lemmas. Although the first of these lemmas is needed only for positive functions, we state and prove it (see appendix) for arbitrary complex-valued functions.

<u>Lemma 2</u>: Let $f \in L_0^2(\mathbb{R})$, $I(f) = [a_f, b_f]$ and $I(\text{auto}(f)) = [-d, d]$. Then

$$d = b_f - a_f.$$

The next lemma is really the heart of the matter.

Lemma 3: Let $f \in L_0^2(\mathfrak{R})$, $f \geq 0$ and $I(\text{auto}(f)) = [-d, d]$. If $\gamma > 0$ and

6.2) $$S(\text{auto}(f)) \subseteq [-d, -\tfrac{1}{2}d] \cup (-\gamma, \gamma) \cup [\tfrac{1}{2}d, d]$$

then

6.3 $$S(f) \subseteq [a_f, a_f + \gamma) \cup (b_f - \gamma, b_f].$$

Note: Lemma 3 is not true if the condition that $f \geq 0$ is dropped. The function described in Figure 5 is a counterexample for $\gamma = d$.

Now suppose f satisfies the hypothesis of Lemma 3 with $\gamma = d/3$. Let

$$f_1(x) = \begin{cases} f(x) \text{ for } x \in [a_f, a_f + \frac{1}{3}d) \\ 0 \text{ otherwise} \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x) \text{ for } x \in (b_f - \frac{1}{3}d, b_f] \\ 0 \quad \text{otherwise} \end{cases}$$

Using this notation, we have the following theorem.

Theorem 9: Suppose $f \in L_0^2(\mathfrak{R})$, $f \geq 0$ and $I(\text{auto}(f)) = [-d, d]$. If

$$S(\text{auto}(f)) \subseteq [-d, -\tfrac{1}{2}d] \cup (-\tfrac{1}{3}d, \tfrac{1}{3}d) \cup [\tfrac{1}{2}d, d]$$

and

$$Z(F_1) \cap Z(F_2) = \emptyset,$$

then f is unique among nonnegative functions; i.e., if $g \in L_0^2(\mathfrak{R})$, $g \geq 0$ and

$$G(u) = F(u) \quad \forall \; u \in \mathfrak{R}$$

then $g = f$.

Of course, using this theorem requires the a priori knowledge that the function being searched for is positive. However, there are many applications in which this is given.

We now turn our attention to the case in which G is gotten from F by flipping only a finite number of non-real zeroes. Let $\{w_n\}_{n=1}^{N}$ be an arbitrary finite sequence of distinct non-real zeroes of F and let

$$G(w) = F(w) \prod_{n=1}^{N} \left[\frac{1 - \frac{w}{\bar{w}_n}}{1 - \frac{w}{w_n}} \right]^{\beta_n}$$

where the β_n are integers satisfying $0 < \beta_n \leq n_F(w_n)$ and F is the Laplace transform of a complex-valued function $f \in L_0^2(\mathfrak{R})$. Let g be the inverse transform of G. Then, by Theorem 7, $g \in L_0^2(\mathfrak{R})$ and $I(f) = I(g)$.

Let $S(f)'$ denote the complement of $S(f)$ with respect to $\mathfrak{R}$. Let

$$S(f)' = (-\infty, a_f) \cup \left[\bigcup_{m=1}^{M} P_m \right] \cup (b_f, \infty)$$

be the decomposition of the open set $S(f)'$ into disjoint open intervals where M is either a finite integer or $M = \infty$. Similarly, let

$$S(g)' = (-\infty, a_f) \cup \left[\bigcup_{k=1}^{K} Q_k \right] \cup (b_f, \infty)$$

in the decomposition of f [illegible] where k is either a finite integer or [illegible]. Recall that, by Theorem [illegible], $a_g = a_f$ and $b_g = b_f$. Finally let

$$P_m = (s_m, t_m), \quad m = 1, \ldots, M.$$

<u>Theorem [illegible]</u>: Assume the conditions defined in the preceding two paragraphs. If for some m and k, $P_m \cap Q_k \neq \emptyset$, then,

(i) $s_m = [illegible]_k$

and

(ii) If

$$f_{-1}(x) = \begin{cases} f(x) \text{ for } a_f \leq x \leq s_m \\ 0 \quad \text{otherwise} \end{cases}$$

$$f_1(x) = \begin{cases} f(x) \text{ for } t_m \leq x \leq b_f \\ 0 \quad \text{otherwise} \end{cases}$$

$$g_{-1}(x) = \begin{cases} g(x) \text{ for } a_f \leq x \leq s_m \\ 0 \quad \text{otherwise} \end{cases}$$

$$g_1(x) = \begin{cases} g(x) \text{ for } t_m \leq x \leq b_f \\ 0 \quad \text{otherwise} \end{cases}$$

then the w_n, n = 1, . . ., N, are zeroes of both F_{-1} and F_1 of order $\geq \beta_n$ and

$$G_r(w) = F_r(w) \prod_{n=1}^{N} \left[\frac{1 - \frac{w}{\bar{w}_n}}{1 - \frac{w}{w_n}} \right]^{\beta_n}, \quad r = -1, 1.$$

This result was proved by Greenaway [7] for the case in which $r_f = 1$ and Im $(w_n) > 0$, $n = 1, \ldots, N$. Example [illegible] shows that this theorem does not hold when an infinite number of zeroes are flipped.

Next, we consider functions $f \in L^2_c(\mathfrak{R})$ for which $W(F)$ (see Section [illegible] for definition) is a finite set. In particular, this condition is satisfied if F has only a finite number of non-real zeroes. Theorems [illegible] and 11 show that the supports of such functions and the functions themselves satisfy some special conditions.

For simplicity of discussion, it will be assumed that $c_f = 0$. The open intervals $P_m = (s_m, t_m)$, $m = 1, \ldots, M$, are defined as above.

<u>Theorem 11</u>: Assume $c_f = 0$. If $W(F)$ is a finite set and some m_0, [illegible] P_{m_0}, then:

(i) $P_{m_0} = -P_{m_0}$

(ii) If

$$f_{-1}(x) = \begin{cases} f(x) \text{ for } x < 0 \\ 0 \text{ otherwise} \end{cases}$$

$$f_1(x) = \begin{cases} f(x) \text{ for } x > 0 \\ 0 \text{ otherwise} \end{cases}$$

then each $z \in W(F)$ is a zero of both F_{-1} and F_1 of orders $\geq \sigma_F(z)$ (see Section [illegible] for definition of σ_F) and

$$F^*_{-1}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} F_1(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)}$$

where $r = 2n_F(0)$.

Theorem 1[illegible]: Assume $c_f = 0$. If $W(F)$ is finite and for some m_1 and m_2, $s_{m_2} > 0$ and $P_{m_1} \cap (-P_{m_2}) \neq \emptyset$ then;

(i) $P_{m_1} = -P_{m_2}$

(ii) If

$$f_{-1}(x) = \begin{cases} f(x) \text{ for } x \leq -t_{m_2} \\ 0 \text{ otherwise} \end{cases}$$

$$f_0(x) = \begin{cases} f(x) \text{ for } -s_{m_2} \leq x \leq s_{m_2} \\ 0 \text{ otherwise} \end{cases}$$

$$f_1(x) = \begin{cases} f(x) \text{ for } x \geq t_{m_2} \\ 0 \text{ otherwise} \end{cases}$$

then each $z \varepsilon W(F)$ is a zero of F_{-1}, F_0 and F_1 of orders $\geq \sigma_F(z)$,

$$F^*_{-1}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} F_1(w) \prod_{z \varepsilon W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)}$$

and

$$F^*_0(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} F_0(w) \prod_{z \varepsilon W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)}$$

where $r = 2n_F(0)$.

[illegible]. Summary

The problem of uniqueness in phase retrieval was explored. Methods of generating unique and non-unique functions were presented. It was shown that if a function is comprised of parts within regions of support satisfying the separation condition of Eq. 6.1[illegible] and if there are no non-real zeros common to all the transforms of the segments, then the function is unique among functions with the same support; however, the funtion is not necessarily unique among all functions. Furthermore, it was shown that if it is known that one function is gotten from the other by flipping at most a finite number of zeros, then the same conclusion holds with the condition of Eq. 6.[illegible] replaced by the weaker condition that the regions simply be disjoint. For nonnegative functions having autocorrelations whose supports satisfy certain disconnection conditions, it was shown that if the transforms of the segments of the function have no non-real zeros in common, then the function is unique among nonnegative functions.

Finally, it was shown that if the transform of a function has only a finite number of non-real zeros, then the support of the function as well as the function itself satisfy some special conditions.

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8. Appendix

Proof of Theorem 1: We have

$$G(w) = \sum_{n=0}^{N} c_n\, e^{-in\beta w} F(w)$$

$$= F(w) \sum_{n=0}^{N} c_n (e^{-i\beta w})^n$$

$$= F(w)\alpha \prod_{n=1}^{N} (e^{-i\beta w} - b_n).$$

Also,

$$H(w) = \sum_{n=0}^{N} d_n\, e^{-in\beta w} F(w)$$

$$= F(w) \sum_{n=0}^{N} d_n (e^{-i\beta w})^n$$

$$= F(w)\alpha \left[\prod_{n\epsilon B} (-\bar{b}_n\, e^{-i\beta w} + 1)\right] \cdot \left[\prod_{n\epsilon C} (e^{-i\beta w} - b_n)\right]$$

$$= G(w)\varphi(w)$$

where

$$\varphi(w) = \prod_{n\epsilon B} \left(\frac{-\bar{b}_n\, e^{-i\beta w} + 1}{e^{-i\beta w} - b_n}\right).$$

For $u \epsilon \Re$,

$$\left|\frac{-\bar{b}_n e^{-i\beta u} + 1}{e^{-i\beta u} - b_n}\right|^2 = \frac{(-\bar{b}_n\, e^{-i\beta u} + 1)\ \ (-b_n\, e^{i\beta u} + 1)}{(e^{-i\beta u} - b_n)\ \ (e^{i\beta u} - \bar{b}_n)}$$

$$= \frac{|b_n|^2 - 2\ \mathrm{Re}\ (b_n\, e^{i\beta u}) + 1}{1 - 2\ \mathrm{Re}\ (b_n\, e^{i\beta u}) + |b_n|^2}$$

$$= 1.$$

Therefore, for $u \in \mathfrak{R}$, $|\varphi(u)| = 1$ and

$$|H(u)| = |G(u)| \cdot |\varphi(u)| = |G(u)| . \qquad \text{Q.E.D.}$$

Proof of Theorem 2: We have

$$H_1(w) = F(w)G(w)$$

and

$$H_2(w) = F(w)G^*(w).$$

For $u \in \mathfrak{R}$,

$$G^*(u) = \overline{G(u)}$$

and therefore

$$|G^*(u)| = |G(u)| .$$

Thus,

$$|H_1(u)| = |F(u)| \cdot |G(u)|$$

$$= |F(u)| \cdot |G^*(u)|$$

$$= |H_2(u)| . \qquad \text{Q.E.D.}$$

Proofs of Theorems 3 and 4: The proofs of these theorems and the corollary to Theorem 3 use Corollary 3 to Theorem 8 and therefore appear immediately following the proof of that corollary.

Proof of Theorem 5: First, suppose $m = 0$. Then $F(0) \neq 0$ and since $f \in L_0^2(\mathfrak{R})$, f is integrable. Therefore, by [3], p. 28[illegible], Theorem VI,

$$F(w) = F(0)\, e^{-ic_f w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{\overline{w}_n}\right)^{n_F(\overline{w}_n)}$$

Now suppose $m > 0$. Let

$$g_1(x) = \int_{a_f}^{x} f(t)\, dt$$

Then $a_{g_1} = a_f$, $b_{g_1} = b_f$ and $c_{g_1} = c_f$. Also,

$$\begin{aligned} \int_{a_f}^{b_f} |g_1(x)|\, dx &= \int_{a_f}^{b_f} \left| \int_{a_f}^{x} f(t)\, dt \right| dx \\ &\leq \int_{a_f}^{b_f} \int_{a_f}^{x} |f(t)|\, dt\, dx \\ &\leq \int_{a_f}^{b_f} \int_{a_f}^{b_f} |f(t)|\, dt\, dx \\ &= (b_f - a_f) \int_{a_f}^{b_f} |f(t)|\, dt < \infty. \end{aligned}$$

Therefore g is integrable. Furthermore

$$F(w) = \int_{a_f}^{b_f} f(x)\, e^{-iwx}\, dx$$

$$= g_1(x)\, e^{-iwx} \Big|_{a_f}^{b_f} + iw \int_{a_f}^{b_f} g_1(x)\, e^{-iwx}\, dx$$

$$= iwG_1(w).$$

It follows that $\eta_{G_1}(w) = \eta_F(w) \;\forall\; w \neq 0$. By repeating this process m times, we obtain

$$F(w) = (iw)^m G_m(w)$$

where $G_m(0) \neq 0$. Now we apply Titchmarsh's theorem to G_m and obtain

$$F(w) = Aw^m\, e^{-ic_f w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right)^{\eta_F(w_n)} \left(1 - \frac{w}{\bar{w}_n}\right)^{\eta_F(\bar{w}_n)}$$

where A is a constant.

To evaluate A, let

$$\varphi(w) = \frac{1}{w^m} F(w).$$

Then, by expanding F in a Maclaurin series, we obtain

$$\varphi(0) = \frac{1}{m!} F^{(m)}(0).$$

On the other hand,

$$\varphi(w) = A\, e^{-ic_f w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right)^{\eta_F(w_n)} \left(1 - \frac{w}{\bar{w}_n}\right)^{\eta_F(\bar{w}_n)}$$

Setting $w = 0$, we obtain

$$G(0) = A.$$

Therefore

$$A = \frac{1}{m!} F^{(m)}(0). \qquad \text{Q.E.D.}$$

<u>Proof of Theorem 6</u>: Let $\gamma_F(w) = \eta_F(w)$ for $w \in \mathfrak{C} \setminus \mathfrak{R}$ and $\gamma_F(u) = 2\eta_F(u)$ for $u \in \mathfrak{R}$. By the Paley-Weiner Theorem ([2], p. 103), $F(w)$ is an entire function of exponential type and by the Hadamard Factorization Theorem ([2], p. 22)

$$F(w) = Aw^m e^{aw} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right)^{\gamma_F(w_n)} \exp\left[\frac{w}{w_n} \gamma_F(w_n)\right]$$

where the infinite product is absolutely convergent for all $w \in \mathfrak{C}$. By [3], p. 284, Theorem II, the series

$$\sum_{n=1}^{\infty} \gamma_F(w_n) \operatorname{Im}\left(\frac{1}{w_n}\right)$$

is absolutely convergent.

It then follows (see [8], Theorem 8, p. 223) that the product

$$\prod_{n=1}^{\infty} \exp\left[iw\gamma_F(w_n) \operatorname{Im}\left(\frac{1}{w_n}\right)\right]$$

is absolutely convergent. Therefore, the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right)^{\gamma_F(w_n)} \exp\left[w\gamma_F(w_n) \operatorname{Re}\left(\frac{1}{w_n}\right)\right]$$

is absolutely convergent. Now $F^*(w)$ is also an entire function of exponential type with zeroes $\overline{w}_n$ of orders $\gamma_{F^*}(\overline{w}_n) = \gamma_F(w_n)$. Therefore, by the same reasoning,

$$\prod_{n=1}^{\infty}\left(1-\frac{w}{\bar{w}_n}\right)^{\gamma_{F^*}(\bar{w}_n)} \exp\left[w\gamma_{F^*}(\bar{w}_n)\ \mathrm{Re}\left(\frac{1}{\bar{w}_n}\right)\right]$$

$$=\prod_{n=1}^{\infty}\left(1-\frac{w}{\bar{w}_n}\right)^{\gamma_F(w_n)} \exp\left[w\gamma_F(w_n)\ \mathrm{Re}\left(\frac{1}{w_n}\right)\right]$$

converges absolutely. Then

$$\prod_{n=1}^{\infty}\left[\frac{1-\dfrac{w}{\bar{w}_n}}{1-\dfrac{w}{w_n}}\right]^{\gamma_F(w_n)}$$

$$=\prod_{n=1}^{\infty}\frac{\left(1-\dfrac{w}{\bar{w}_n}\right)^{\gamma_F(w_n)} \exp\left[w\gamma_F(w_n)\ \mathrm{Re}\left(\dfrac{1}{w_n}\right)\right]}{\left(1-\dfrac{w}{w_n}\right)^{\gamma_F(w_n)} \exp\left[w\gamma_F(w_n)\ \mathrm{Re}\left(\dfrac{1}{w_n}\right)\right]}$$

converges absolutely. It follows a fortiori that

$$\prod_{n=1}^{\infty}\left[\frac{1-\dfrac{w}{\bar{w}_n}}{1-\dfrac{w}{w_n}}\right]^{\beta(w_n)}$$

converges absolutely. Q.E.D.

Proof of Theorem 7: Let

$$\varphi(w) = \prod_{n=1}^{\infty} \left[\frac{1 - \frac{w}{\bar{w}_n}}{1 - \frac{w}{w_n}} \right]^{\beta(w_n)}$$

By Theorem 6, this infinite product is absolutely convergent $\forall$ $w \in \mathbb{C}$. Since

$$|\varphi(u)| = 1 \;\forall\; u \in \mathbb{R},$$

$|G(u)| = |F(u)|$. Since $F(u)$ is square-summable, it follows that G and g are square-summable. It remains to show

$$I(g) = I(f).$$

First we shall show that $I(g) \subseteq I(f)$. Let

$$\varphi_m(w) = \prod_{n=1}^{m} \left[\frac{1 - \frac{w}{\bar{w}_n}}{1 - \frac{w}{w_n}} \right]^{\beta(w_n)},$$

$$G_m(w) = \varphi_m(w) F(w)$$

and let g_m be the inverse transform of G_m.

Claim

(8.1) $$I(g_m) \subseteq I(f).$$

Let w_0 be an arbitrary non-real zero of F. Let

$$H(w) = \frac{1 - \frac{w}{\bar{w}_0}}{1 - \frac{w}{w_0}} F(w)$$

and let h be the inverse transform of H. To prove 8.1), it suffices to show that

8.2) $$I(h) \subseteq I(f).$$

Let

$$\Gamma(w) = \frac{1}{w_0 - w} .$$

Then $\Gamma(w)$, $w \in \mathbb{R}$, is square-summable,

$$\frac{1 - \frac{w}{\bar{w}_0}}{1 - \frac{w}{w_0}} = \frac{w_0}{\bar{w}_0} - \frac{w_0}{\bar{w}_0}(w_0 - \bar{w}_0)\,\Gamma(w)$$

and

8.3) $$H(w) = \frac{w_0}{\bar{w}_0} F(w) - \frac{w_0}{\bar{w}_0}(w_0 - \bar{w}_0)\,\Gamma(w)F(w).$$

Let γ be the inverse transform of Γ. Then

$$\gamma(x) = -\frac{i}{2}\left[\operatorname{sgn}(v_0) + \operatorname{sgn}(x)\right] e^{iw_0 x}$$

where $w_0 = u_0 + iv_0$ and

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases}$$

From 8.3), we get

$$h(x) = \frac{w_0}{\bar{w}_0} f(x) - \frac{w_0}{\bar{w}_0}(w_0 - \bar{w}_0)(\gamma * f)(x)$$

To prove 8.2), it suffices to show that

8.4) $$I(\gamma * f) \subseteq I(f).$$

Now

$$(\gamma * f)(x) = -\frac{i}{2}\int_{-\infty}^{\infty} f(y)[\operatorname{sgn}(v_0) + \operatorname{sgn}(x-y)]\, e^{iw_0(x-y)}\, dy.$$

First suppose $v_0 > 0$. Then

$$(\gamma * f)(x) = -\frac{i}{2}\, e^{iw_0x}\int_{-\infty}^{x} f(y)\, e^{-iw_0y}\, dy$$

Therefore, for $x < a_f$, $(\gamma * f)(x) = 0$. For $x > b_f$

$$\begin{aligned}(\gamma * f)(x) &= -\frac{i}{2}\, e^{iw_0x}\int_{-\infty}^{x} f(y)\, e^{-iw_0y}\, dy \\ &= -\frac{i}{2}\, e^{iw_0x}\int_{-\infty}^{\infty} f(y)\, e^{-iw_0y}\, dy \\ &= -\frac{i}{2}\, e^{iw_0x} F(w_0) \\ &= 0.\end{aligned}$$

A similar argument yields the same result for $v_0 < 0$. This proves 8.4 (which implies 8.2) which in turn implies 8.1).

We have

$$|G(u) - G_m(u)|^2 \xrightarrow[m\to\infty]{} 0 \;\forall\; u \in \mathfrak{R}.$$

Furthermore, since $|\varphi(u)| = |\varphi_m(u)| = 1$,

$$\begin{aligned}|G(u) - G_m(u)|^2 &= |\varphi(u)F(u) - \varphi_m(u)F(u)|^2 \\ &= |\varphi(u) - \varphi_m(u)|^2 \cdot |F(u)|^2 \\ &\le 4\,|F(u)|^2\end{aligned}$$

Since $|F(u)|^2$ is summable, it follows from the Lebesgue dominated convergence theorem that

$$\int_{-\infty}^{\infty} |G(u) - G_m(u)|^2 \, du \xrightarrow[m\to\infty]{} 0,$$

i.e., G_m converges to G in the L^2 norm. Therefore g_m converges to g in the L^2 norm. Since

$$I(g_m) \subseteq I(f), \quad m = 1, 2, 3, \ldots,$$

it follows that

$$I(g) \subseteq I(f).$$

Now let $\alpha(w) = \beta(\bar{w})$. We have

$$n_G(w) = n_F(w) - \beta(w) + \beta(\bar{w}).$$

Therefore $0 \leq \alpha(w) \leq n_G(w)$ for $w \in \mathbb{C} \sim \mathfrak{R}$ and $0 \leq \alpha(u) \leq 2n_G(u)$ for $u \in \mathfrak{R}$. Let $\{z_n\}_{n=1}^{\infty}$ be the distinct non-zero zeroes of G. Then

$$F(w) = G(w) \prod_{n=1}^{\infty} \left[\frac{1 - \frac{w}{\bar{z}_n}}{1 - \frac{w}{z_n}} \right]^{\alpha(z_n)}$$

and a similar argument yields

$$I(f) \subseteq I(g) \qquad \text{Q.E.D.}$$

Proof of Theorem 8: Assume $|F(u)| = |G(u)|$ $\forall$ $u \in \mathfrak{R}$. First we consider the case $N \neq 2$. Then $f * \tilde{f} = g * \tilde{g}$. For $n \neq m$, $f_n * \tilde{f}_m$ is the restriction of $f * \tilde{f}$ to $I_n - I_m$ and $g_n * \tilde{g}_m$ is the restriction of $g * \tilde{g}$ to $I_n - I_m$. Therefore

$$f_n * \tilde{f}_m = g_n * \tilde{g}_m$$

and

$$F_n F_m^* = G_n G_m^* \text{ for } n \neq m$$

If $N \geq 3$, let n_1, n_2 and n_3 be three distinct integers ≥ 1 and $\leq N$. Then

$$F_{n_1} F^*_{n_2} F_{n_2} F^*_{n_3} = G_{n_1} G^*_{n_2} G_{n_2} G^*_{n_3} \tag{8.6}$$

and

$$F_{n_1} F^*_{n_3} = G_{n_1} G^*_{n_3}. \tag{8.7}$$

Since all of these functions are entire, we may divide Equation 8.6 by Equation 8.7 and obtain

$$F^*_{n_2} F_{n_2} = G^*_{n_2} G_{n_2}.$$

Therefore, if $N \geq 3$,

$$F_n F^*_n = G_n G^*_n \text{ for } n = 1, \ldots, N \tag{8.8}$$

Equation 8.8 holds also for $N = 1$ since $FF^* = GG^*$ and, in this case, $F_1 = F$ and $G_1 = G$. Putting 8.5 and 8.8 together;

$$F_n F^*_m = G_n G^*_m \text{ for } n, m = 1, \ldots, N. \tag{8.9}$$

It follows from 8.9 that

$$D(F_n) = D(G_n)$$

and [illegible], $n = 1, \ldots, N$. Now we number the elements of [illegible],

$$D(F_k) = \{w_{k,n}: n = 1, 2, \ldots\},$$

such that the sequence $\{|w_{k,n}|\}_{n=1}^{\infty}$ is non-decreasing.

Let [illegible]. Then, by Theorem [illegible],

8.10)

$$F_k(w) = \frac{1}{b_k!} F_k^{(b_k)}(0) w^{b_k} e^{-ic_{f_k} w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_{k,n}}\right)^{\eta_k(w_{k,n})} \left(1 - \frac{w}{\bar{w}_{k,n}}\right)^{\eta_k(\bar{w}_{k,n})}$$

and

8.11)

$$G_k(w) = \frac{1}{b_k!} G_k^{(b_k)}(0) w^{b_k} e^{-ic_{g_k} w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_{k,n}}\right)^{\zeta_k(w_{k,n})} \left(1 - \frac{w}{\bar{w}_{k,n}}\right)^{\zeta_k(\bar{w}_{k,n})} .$$

Now from 8.9), we obtain

$$\eta_n(w) + \eta_m(\bar{w}) = \zeta_n(w) + \zeta_m(\bar{w})$$

and hence

8.12)

$$\eta_n(w) - \zeta_n(w) = \zeta_m(\bar{w}) - \eta_m(\bar{w})$$

for $n, m = 1, \ldots, N$ and $\forall\, w \in \mathbb{C}$. It follows from 8.12) that the left side of this equation is constant with respect to n and the right side is constant with respect to m. Let

$$\beta(w) = \eta_n(w) - \zeta_n(w).$$

Then

$$\beta(\bar{w}) = -\beta(w).$$

Let

$$\alpha(w) = \begin{cases} \beta(w) \text{ when } \beta(w) > 0 \\ 0 \text{ otherwise} \end{cases}$$

Then

8.13)

$$0 \leq \alpha(w) \leq \min_{1 \leq n \leq N} \eta_n(w)$$

from which it follows that

(3.14) $\qquad \alpha(w) = 0$ for $w \notin B$.

[illegible]

$$\phi(w) = \prod_{z \in B} \left[\frac{1 - \frac{w}{z}}{1 - \frac{w}{\bar{z}}}\right]^{\alpha(z)} \tag{3.15}$$

[illegible] Theorem [illegible], the (possibly infinite) product in (3.15) is absolutely convergent and hence its value is independent of the order of the factors. Therefore $\phi(w)$ is well-defined.

[illegible] (3.14) [illegible] can also be expressed, for any fixed k, $1 \le k \le N$, as

$$\phi(w) = \prod_{n=1}^{\infty} \left[\frac{1 - \frac{w}{w_{k,n}}}{1 - \frac{w}{\bar{w}_{k,n}}}\right]^{\alpha(w_{k,n})} \left[\frac{1 - \frac{w}{\overline{w_{k,n}}}}{1 - \frac{w}{\bar{w}_{k,n}}}\right]^{\alpha(\overline{w_{k,n}})}$$

Now

$$[\text{illegible}]^{\alpha(w_{k,n})} \cdot \left[\frac{1 - \frac{w}{w_{k,n}}}{1 - \frac{w}{\bar{w}_{k,n}}}\right]^{\alpha(\bar{w}_{k,n})}$$

$$\cdot \left(1 - \frac{w}{w_{k,n}}\right)^{\alpha_k(w_{k,n})} \cdot \left(1 - \frac{w}{\bar{w}_{k,n}}\right)^{\alpha_k(\bar{w}_{k,n})}$$

$$= \left(1 - \frac{w}{w_{k,n}}\right)^{\alpha_k(w_{k,n})} \left(1 - \frac{w}{\bar{w}_{k,n}}\right)^{\alpha_k(\bar{w}_{k,n})}$$

Therefore,

[illegible]

$$[illegible] \prod_{n=1}^{[illegible]} \left(1 - \frac{w}{w_{k,n}}\right)^{n_k(w_{k,n})} \left(1 - \frac{w}{\bar{w}_{k,n}}\right)^{n_k(\bar{w}_{k,n})}$$

$$= \prod_{n=1}^{[illegible]} \left(1 - \frac{w}{w_{k,n}}\right)^{[illegible]_k(w_{k,n})} \left(1 - \frac{w}{\bar{w}_{k,n}}\right)^{[illegible]_k(\bar{w}_{k,n})} .$$

[illegible]

[illegible] $$A_k = \frac{G_k^{(m_k)}(0)}{F_k^{(m_k)}(0)} .$$

Then by [illegible].10), [illegible].11), 8.17), and 8.18),

[illegible] $$G_k(w) = A_k\, e^{id_k w}\,[illegible](w)F_k(w)$$

where $d_k = c_{[illegible]} - [illegible]_{g_k}$.

Now

$$[illegible]^{\star}(w) = \prod_{z \in [illegible]} \left[\frac{1 - \frac{w}{z}}{1 - \frac{w}{\bar{z}}}\right]^{\alpha(z)}$$

Therefore, by [illegible]) and [illegible].20),

$$[illegible](w)\,[illegible]^{\star}(w) = 1 \quad \forall\; w \in \mathbb{C}$$

[illegible],

$$F_n(w)F_m^{\star}(w) = G_n(w)G_m^{\star}(w)$$

$$= A_n\bar{A}_m\, e^{i(d_n - d_m)w} F_n(w)F_m^{\star}(w)$$

for $n, m = 1, \ldots, N$ and $\forall\, w \in \mathbb{C}$.

Therefore,

[illegible] $$\alpha_n \bar{\alpha}_m e^{i(d_n - d_m)w} = 1 \quad \forall\, w \in \mathbb{C}.$$

It follows that

[illegible] $$d_n = d_m, \quad n, m = 1, \ldots, N.$$

Also, setting $n = m$ in (8.2?), we obtain

$$|\alpha_n|^2 = 1, \quad n = 1, \ldots, N.$$

Let $\alpha_n = e^{i\theta_n}$. Then from (8.2?) and (8.22),

$$e^{i(\theta_n - \theta_m)} = 1$$

and therefore

$$\theta_n = \theta_m, \quad n, m = 1, \ldots, N.$$

Let $\theta = \theta_n$ and $d = d_n$, $n = 1, \ldots, N$. Then by (8.19)

$$G_n(w) = e^{idw} e^{i\theta} c(w) F_n(w)$$

$$= e^{idw} c'(w) F_n(w), \quad n = 1, \ldots, N.$$

where $c'(w) = e^{i\theta} c(w)$.

It remains to show that $d = c_f - c_g$. From (8.2?), we obtain

$$G(w) = e^{idw} c'(w) F(w).$$

From Theorems 6 and 7, it follows that $d = c_f - c_g$.

Now consider the case $N = 2$. Let

$$I = (I_1 - I_1) \cup (I_2 - I_2).$$

It follows from (5.1) that

(8.24) $$(I \cap I_n - I_m) = \emptyset \text{ for } n \neq m.$$

where $F_1F_1^* + F_2F_2^*$ is the restriction of FF^* to I and $G_1G_1^* + G_2G_2^*$ is the restriction of GG^* to I. Therefore, since $FF^* = GG^*$,

(8.25) $$F_1F_1^* + F_2F_2^* = G_1G_1^* + G_2G_2^* .$$

Therefore, we also have

(8.26) $$F_1F_2^* = G_1G_2^* \text{ and } F_1^*F_2 = G_1^*G_2$$

From (8.25) and (8.26), we obtain

(8.27) $$\begin{aligned}(F_1F_1^* - G_1G_1^*)\,(F_1F_1^* - G_2G_2^*) &= (F_1F_1^*)^2 - (G_1G_1^* + G_2G_2^*)F_1F_1^* + (G_1G_2^*)(G_1^*G_2) \\ &= (F_1F_1^*)^2 - (F_1F_1^* + F_2F_2^*)F_1F_1^* + (F_1F_2^*)(F_1^*F_2) \\ &= 0.\end{aligned}$$

Therefore, since all functions involved are entire, either

(8.28) $$F_1F_1^* = G_1G_1^* \text{ and, by (8.25), } F_2F_2^* = G_2G_2^*$$

or

(8.29) $$F_1F_1^* = G_2G_2^* \text{ and, by (8.25), } F_2F_2^* = G_1G_1^*.$$

If (8.28) holds, then the same argument used in the case $N \neq 2$ applies and conclusion 1) of the theorem follows. If (8.29) holds, let

$$H_1 = G_2^*,\ H_2 = G_1^* \text{ and } H = H_1 + H_2.$$

Then

$$H_1H_1^* = F_1F_1^*, \quad H_2H_2^* = F_2F_2^*$$

$$H_1H_2^* = F_1F_2^*, \quad H_1^*H_2 = F_1^*F_2$$

and

$$HH^* = FF^*.$$

Therefore the argument used in the case $N \neq 2$ applies with G, G_1 and G_2 replaced by H, H_1 and H_2 respectively and Conclusion 1) of the theorem follows.

Now assume that Conclusion 1) of the theorem holds. Then

$$G(w) = e^{i(c_f-c_g)w}\varphi(w)F(w)$$

and for $u \in \mathfrak{R}$,

$$\begin{aligned} |G(u)|^2 &= G(u)G^*(u) = \varphi(u)\varphi^*(u)F(u)F^*(u) \\ &= F(u)F^*(u) \\ &= |F(u)|^2. \end{aligned}$$

If Conclusion 2) of the theorem holds, then

$$G^*(w) = e^{i(c_f+c_g)}\varphi(w)F(w)$$

and for $u \in \mathfrak{R}$,

$$\begin{aligned} |G(u)|^2 &= G(u)G^*(u) = \varphi^*(u)\varphi(u)F^*(u)F(u) \\ &= F^*(u)F(u) \\ &= |F(u)|^2. \end{aligned}$$

Q.E.D.

<u>Proof of Corollary 1 to Theorem 8</u>: The implication

$$f \sim g \Rightarrow |F(u)| = |G(u)| \quad \forall\, u \in \mathfrak{R}$$

is immediate.

Now assume $|F(u)| = |G(u)|$ $\forall$ $u \in \Re$. Since $B = \phi$,

$$\epsilon(w) = e^{i\theta}.$$

If conclusion I) of Theorem 8 holds, then

$$G(w) = e^{i\theta} e^{i(c_f - c_g)w} F(w).$$

Therefore

$$g(x) = e^{i\theta} f(x + c_f - c_g)$$

and $f \sim g$.

If conclusion II) holds, then

$$G^*(w) = e^{i\theta} e^{i(c_f + c_g)w} F(w).$$

Therefore

$$\tilde{g}(x) = e^{i\theta} f(x + c_f + c_g)$$

and $f \sim g$. Q.E.D.

Proof of Corollary 2 to Theorem 8: Here, $N = 1$ and $B = Z(F)$. The corollary follows. Q.E.D.

Proof of Lemma 1: By Theorem 5,

$$F(w) = \frac{1}{m!} F^{(m)}(0) w^m e^{-ic_f w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(\bar{w}_n)}$$

Therefore

$$F^*(w) = \frac{1}{m!} \overline{F^{(m)}(0)} w^m e^{ic_f w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{w_n}\right)^{n_F(\bar{w}_n)}.$$

The lemma follows. Q.E.D.

<u>Proof of Corollary 3 to Theorem 8</u>: The equivalence of 2) and 3) is immediate.

<u>1) $\Rightarrow$ 2)</u>: If $F(w)$ has more than one non-real zero, then flipping one of them yields a function $G(w)$ such that, by Corollary 2,

$$G(u) = F(u) \quad \forall\, u \in \mathfrak{R}$$

but f and g are not equivalent.

<u>2) $\Rightarrow$ 1)</u>: By Lemma 1, if $F(w)$ has only one non-real zero of order 1, then flipping this zero yields a function equivalent to $\tilde{f}$. Since $\tilde{f}$ is, in turn, equivalent to f, the implication follows. Q.E.D.

<u>Proof of Theorem 3</u>: We have

$$\begin{aligned} G(w) &= \sum_{n=0}^{N} c_n e^{-i\beta n w} F(w) \\ &= F(w) \sum_{n=0}^{N} c_n \left(e^{-i\beta w}\right)^n \\ &= F(w) \prod_{n=1}^{N} \left(e^{-i\beta w} - b_n\right). \end{aligned}$$

Let

$$\varphi(w) = \prod_{n=1}^{N} \left(e^{-i\beta w} - b_n\right).$$

Then $G(w) = F(w)\varphi(w)$. By Corollary 3 to Theorem 8, since f is unique, F has either no non-real zeroes or one non-real zero of order 1. Therefore, if φ has no non-real zeroes, then G has either no non-real zeroes or one non-real zero of order 1 and, by the same corollary, g is unique. Therefore, it suffices to show that φ has no non-real zeroes. Suppose $\varphi(w_0) = 0$. Then for some n_0, $1 \le n_0 \le N$,

$$e^{-i\beta w_0} = t_{n_0}.$$

and $w_0 = u_0 + iv_0$. Then

$$e^{-i\beta u_0} e^{\beta v_0} = t_{n_0}$$

so

$$e^{\beta v_0} = |t_{n_0}| = 1.$$

Therefore $v_0 =$ [illegible]. Q.E.D.

<u>Proof of Corollary to Theorem 3</u>: We have

$$\sum_{n=0}^{N} x^n = \prod_{n=1}^{N} \left(x - e^{2\pi i \frac{n}{N+1}}\right).$$

Now apply Theorem 3 with

$$t_n = e^{2\pi i \frac{n}{N+1}}, \quad n = 1, \ldots, N. \qquad \text{Q.E.D.}$$

<u>Proof of Theorem 4</u>: Let $h = f * g$. Then $H(w) = F(w)G(w)$. By Corollary to Theorem 3, F has either no non-real zeroes or one non-real zero of order 1. Since G has no non-real zeroes, H has either no non-real zeroes or one non-real zero of order 1. Therefore, by the same Corollary, h is unique. Q.E.D.

<u>Proof of Lemma 2</u>: The inequality

$$d \leq b_f - a_f$$

is immediate. It remains to show

$$d \geq b_f - a_f$$

Suppose $d < b_f - a_f$. Let $g = \text{auto}(f)$. Then

$$g(y) = \int_{a_f}^{b_f - y} \overline{f(x)} f(x + y)\, dx = 0$$

for almost all $y \geq d$. Let

$$\varphi(x) = \overline{f(x + a_f)}, \quad \psi(x) = f(b_f - x),$$

$$t = x - a_f, \quad s = b_f - a_f - y$$

and

$$\kappa = b_f - a_f - d.$$

Then $\kappa > 0$ and

$$\int_0^s \varphi(t)\psi(s - t)\, dt = 0$$

for a.e. $s \leq \kappa$. Under these conditions, Titchmarsh's Theorem VII in [illegible], p. 286, states that $\exists\ \lambda$ and $\mu \ni \varphi(t) = 0$ a.e. in $(0, \lambda)$, $\psi(t) = 0$ a.e. in $(0, \mu)$ and $\lambda + \mu \geq \kappa$. This last inequality implies that either $\lambda > 0$ or $\mu > 0$ (or both).

Suppose $\lambda > 0$. Then $f(x) = 0$ a.e. in $(a_f, a_f + \lambda)$. This contradicts the assumption that the interval of support of f is $[a_f, b_f]$.

On the other hand, suppose $\mu > 0$. Then $f(x) = 0$ a.e. in $(b_f - \mu, b_f)$, again contradicting the interval of support of f. Q.E.D.

Proof of Lemma 3: If $\gamma > d/2$, then by Lemma 2

$$2\gamma > d = b_f - a_f$$

and

$$a_f + \gamma > b_f - \gamma.$$

Therefore

$$[a_f, a_f + \gamma] \cup [b_f - \gamma, b_f] = [a_f, b_f] = [\text{[illegible]}]$$

and the conclusion of the lemma reduces to the trivial fact that [illegible] $\supseteq$ [illegible].

Now assume $\gamma \leq d/2$.

Let J_0 be the smallest closed interval containing $S(f * \tilde{f}) \cap$ [illegible], let J_1 be the smallest closed interval containing $S(f * \tilde{f}) \cap$ [illegible]. Let J_{-1} be the smallest closed interval containing $S(f * \tilde{f}) \cap$ [illegible]. Let α and β be defined by

$$J_0 = [-\alpha, \alpha]$$

and

$$J_1 = [\beta, d].$$

Then

$$J_{-1} = [-d, -\beta].$$

Also $\alpha < \gamma \leq d/2$ and $\beta \geq d/2$. Furthermore,

(3.[illegible]) $$(f * \tilde{f})(x) = 0 \text{ for } x \in (-\beta, -\alpha) \cup (\alpha, \beta).$$

For $0 < \varepsilon < \beta - \alpha$, define

$$g_\varepsilon(x) = \begin{cases} f(x) \text{ for } x \in (a_f, a_f + \varepsilon) \\ 0 \text{ otherwise} \end{cases}$$

and

$$h_\varepsilon(x) = \begin{cases} f(x) \text{ for } x \in [a_f + \alpha + \varepsilon, a_f + \beta] \\ 0 \text{ otherwise} \end{cases}$$

For $y < \alpha$ or $y > \beta$,

$$h_\delta(x)g_\delta(x - y) = 0 \quad \forall\, x \in \mathfrak{R}.$$

For $y \in [\alpha, \beta]$,

$$\int_{-\infty}^{\infty} h_\delta(x)g_\delta(x - y)\, dx \le \int_{-\infty}^{\infty} f(x)f(x - y)\, dx = (f * \tilde{f})(y) = 0$$

by (4.3[illegible]). Therefore

$$\int_{-\infty}^{\infty} h_\delta(x)g_\delta(x - y)\, dx = 0 \;\forall\; y \in \mathfrak{R}.$$

That is,

$$h_\delta * \tilde{g}_\delta = 0$$

which implies

$$H_\delta\, G_\delta^* = 0.$$

By definition of the interval of support of f, $g_\delta \neq 0$. Therefore $G_\delta \neq 0$ and $G_\delta^* \neq 0$. Since both H_δ and G_δ^* are entire functions, it follows that $H_\delta = 0$ and $h_\delta = 0$. Therefore,

$$f(x) = 0 \text{ for a.a. } x \in [a_f + \alpha + \delta,\, a_f + \beta].$$

Since δ was arbitrary except for the condition $0 < \delta < \beta - \alpha$,

(4.3[illegible]) $$f(x) = 0 \text{ for a.a. } x \in [a_f + \alpha,\, a_f + \beta]$$

Now redefine g_δ and h_δ, for $0 < \delta < \beta - \alpha$, by

$$g_\delta(x) = \begin{cases} f(x) \text{ for } x \in (b_f - \delta,\, b_f) \\ 0 \text{ otherwise} \end{cases}$$

and

$$g_f(x) = \begin{cases} [illegible] & x \in [b_f - \beta, b_f - \alpha] \\ 0 & \text{otherwise} \end{cases}$$

Then for [illegible] $\geq$ [illegible],

$$[illegible]\, g_f(x - y) = 0 \quad \forall x \in \Re.$$

[illegible],

$$\int [illegible]\, g_f(x - y)\, dx \leq \int f(x)\, h(x - y)\, dx = [illegible] \star \tilde{f}(y) = [illegible]$$

[illegible]. Therefore,

$$\int f_f(x)\, g_f(x - y)\, dx = 0 \quad \forall y \in \Re.$$

Then, [illegible] the same reasoning as above,

$$[illegible] = 0 \text{ for a.a. } x \in [b_f - \beta, b_f - \alpha].$$

[illegible] $= b_f - a_f$. Then,

$$b_f - a_f = \sigma \leq 2\beta$$

[illegible]

$$b_f - \beta \leq a_f + \beta.$$

[illegible]

$$[illegible] = b_f - a_f$$

[illegible]

$$a_f + \alpha < b_f - \alpha$$

Therefore, by [illegible] and [illegible],

$$[a_f + \alpha, [illegible] + \beta] \cup [b_f - \beta, b_f - \alpha] = [a_f + \alpha, b_f - \alpha].$$

[illegible], [illegible] and (5.38),

$$[illegible] = 0 \text{ for a.e. } x \in [a_f + \gamma,\ b_f - \gamma].$$

[illegible],

$$S(f) \subseteq [a_f,\ a_f + \gamma) \cup (b_f - \gamma,\ b_f]. \qquad [illegible]$$

Proof of Theorem [illegible]: Since $|G(u)| = |F(u)|$ $\forall\ u \in \mathfrak{R}$,

[illegible]
$$g * \tilde{g} = f * \tilde{f}$$

Therefore, by Lemma [illegible],

$$S(f) \subseteq [a_f,\ a_f + \tfrac{1}{3} d) \cup (b_f - \tfrac{1}{3} d,\ b_f]$$

[illegible]

$$S(g) \subseteq [a_g,\ a_g + \tfrac{1}{3} d) \cup (b_g - \tfrac{1}{3} d,\ b_g].$$

[illegible] Lemma [illegible] and (5.36),

[illegible]
$$b_f - a_f = d = b_g - a_g$$

Let $h(x) = g(x - a_f + a_g)$. Then $a_h = a_f$ and, by (5.37), $b_h = b_f$. Therefore,

$$S(h) \subseteq [a_f,\ a_f + \tfrac{1}{3} d) \cup (b_f - \tfrac{1}{3} d,\ b_f].$$

Now let

[illegible] be the smallest closed interval containing $S(f) \cap [a_f, [illegible] + \frac{1}{3} d)$,

[illegible] be the smallest closed interval containing $S(f) \cap (b_f - \frac{1}{3} d, b_f]$,

[illegible] be the smallest closed interval containing $S(h) \cap [a_f, a_f + \frac{1}{3} d)$,

[illegible] be the smallest closed interval containing $S(h) \cap (b_f - \frac{1}{3} d, b_f]$,

and let

[illegible]

Then

$$S(f) \supseteq I_1 \cup I_2$$

$$S(h) \supseteq I_1 \cup I_2$$

and I_1 and I_2 satisfy the separation condition (6.1). Furthermore,

$$P = Z(F_1) \cap Z(F_2) = \emptyset.$$

Therefore, by Corollary 1 to Theorem 8, $f \sim h$. Since also $h \sim g$, it follows that $f \sim g$. Q.E.D.

<u>Proof of Theorem 10</u>: If for $n \neq n'$, $w_n = \bar{w}_{n'}$ and $\varepsilon_n \geq \varepsilon_{n'}$, then

$$\left[\frac{1 - \dfrac{w}{\bar{w}_n}}{1 - \dfrac{w}{w_n}}\right]^{\varepsilon_n} \left[\frac{1 - \dfrac{w}{\bar{w}_{n'}}}{1 - \dfrac{w}{w_{n'}}}\right]^{\varepsilon_{n'}} = \left[\frac{1 - \dfrac{w}{\bar{w}_n}}{1 - \dfrac{w}{w_n}}\right]^{\varepsilon_n - \varepsilon_{n'}} .$$

By carrying out such cancellations, we may assume without loss of generality that

$$w_n \neq \bar{w}_{n'} \text{ for } n, n' = 1, \ldots, N.$$

Next we shall show that the w_n are zeroes of both F_1 and F_2 of order $\geq \varepsilon_n$. This is equivalent to showing that

$$F_1^{(s)}(w_n) = F_2^{(s)}(w_n) = 0 \text{ for } s = 0, 1, \ldots, \varepsilon_n - 1$$
$$\text{and } n = 1, \ldots, N.$$

We have

$$\prod_{n=1}^{N}\left[\frac{1-\dfrac{w}{\bar{w}_n}}{1-\dfrac{w}{w_n}}\right]^{\beta_n} = \prod_{n=1}^{N}\left(\frac{w_n(w-\bar{w}_n)}{\bar{w}_n(w-w_n)}\right)^{\beta_n}$$

Let

$$p = \prod_{n=1}^{N}\frac{w_n}{\bar{w}_n}$$

and

$$\Phi(w) = \prod_{n=1}^{N}\left(\frac{w-\bar{w}_n}{w-w_n}\right)^{\beta_n}$$

Then

$$G(w) = pF(w)\Phi(w).$$

Now let

$$H(w) = 1 - \Phi(w) = \frac{\prod_{n=1}^{N}(w-w_n)^{\beta_n} - \prod_{n=1}^{N}(w-\bar{w}_n)^{\beta_n}}{\prod_{n=1}^{N}(w-w_n)^{\beta_n}} \quad [illegible]$$

Since the degree of the numerator of H is less than the degree of the denominator, H has a partial fraction decomposition [9], [illegible] of the form

$$H(w) = \sum_{n=1}^{N}\sum_{r=1}^{\beta_n}\frac{c_{n,r}}{(w-w_n)^r} \quad [illegible]$$

where the $c_{n,r}$ are [illegible].

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We pause here to show, for later use, that

8.41) $$C_{n,\beta_n} \neq 0 \text{ for } n = 1, \ldots, N.$$

Let t be an integer $1 \leq t \leq N$. We wish to show that $C_{t,\beta_t} = 0$. By 8.40),

8.42) $$(w - w_t)^{\beta_t} H(w) = C_{t,\beta_t} + C_{t,\beta_t - 1}(w - w_t) + C_{t,\beta_t - 2}(w - w_t)^2$$

$$+ \ldots + C_{t,1}(w - w_t)^{\beta_t - 1}$$

$$+ (w - w_t)^{\beta_t} \sum_{\substack{n=1 \\ n \neq t}}^{N} \sum_{r=1}^{\beta_n} \frac{c_{n,r}}{(w - w_n)^r}$$

By 8.39),

8.43) $$(w - w_t)^{\beta_t} H(w) = \frac{\prod_{n=1}^{N} (w - w_n)^{\beta_n} - \prod_{n=1}^{N} (w - \overline{w}_n)^{\beta_n}}{\prod_{\substack{n=1 \\ n \neq t}}^{N} (w - w_n)^{\beta_n}}$$

Equating the right sides of Equations 8.42) and 8.43) and setting $w = w_t$, we obtain

$$C_{t,\beta_t} = - \frac{\prod_{n=1}^{N} (w_t - \overline{w}_n)^{\beta_n}}{\prod_{\substack{n=1 \\ n \neq t}}^{N} (w_t - w_n)^{\beta_n}} \neq 0.$$

This proves 8.41).

Now by 8.39) and 8.40),

$$\Phi(w) = 1 - \sum_{n=1}^{N} \sum_{r=1}^{\beta_n} \frac{C_{n,r}}{(w - w_n)^r} .$$

Let

$$\Gamma_{n,r}(w) = \frac{1}{(w - w_n)^r} .$$

Then for real u, $\Gamma_{n,r}(u)$ is square-summable. Let $\gamma_{n,r}$ be its inverse transform.

Then

8.44) $$\gamma_{n,r}(x) = \frac{i^r}{2\,(r - 1)!} [\mathrm{sgn}\,(v_n) + \mathrm{sgn}\,(x)] x^{r-1} e^{i w_n x}$$

where $w_n = u_n + iv_n$.

We have

$$\Phi(w) = 1 - \sum_{n=1}^{N} \sum_{r=1}^{\beta_n} C_{n,r}\, \Gamma_{n,r}(w),$$

$$G(w) = pF(w)\Phi(w)$$

$$= pF(w) - p\sum_{n=1}^{N} \sum_{r=1}^{\beta_n} C_{n,r} F(w)\, \Gamma_{n,r}(w),$$

and

8.45) $$g(x) = pf(x) - p\sum_{n=1}^{N} \sum_{r=1}^{\beta_n} C_{n,r} (f * \gamma_{n,r})(x).$$

Let $T = P_m \cap Q_k$. By assumption $T \neq \phi$. Since P_m and Q_k are open intervals, T is an open interval. For $x \epsilon T$, $f(x) = 0$ and $g(x) = 0$. Therefore, from 8.45),

(8.46) $$\sum_{n=1}^{N} \sum_{r=1}^{\beta_n} C_{n,r}(f * \gamma_{n,r})(x) = 0 \text{ for } x \epsilon T.$$

Now

$$(f * \gamma_{n,r})(x) = \int_{-\alpha}^{\alpha} f(y)\gamma_{n,r}(x - y)\, dy$$

$$= \frac{i^r}{2(r-1)!} \int_{-\alpha}^{\alpha} f(y)[\operatorname{sgn}(v_n) + \operatorname{sgn}(x - y)]$$

$$(x - y)^{r-1}\, e^{iw_n(x-y)}\, dy.$$

First assume $v_n > 0$. Then

$$(f * \gamma_{n,r})(x) = \frac{i^r}{(r-1)!} e^{iw_n x} \int_{-\alpha}^{x} f(y)(x - y)^{r-1} e^{-iw_n y}\, dy.$$

If $x \epsilon T$, then

$$(f * \gamma_{n,r})(x) = \frac{i^r}{(r-1)!} e^{iw_n x} \int_{-\alpha}^{x} f_{-1}(y)(x - y)^{r-1} e^{-iw_n y}\, dy$$

$$= \frac{i^r}{(r-1)!} e^{iw_n x} \int_{-\alpha}^{\alpha} f_{-1}(y)(x - y)^{r-1} e^{-iw_n y}\, dy$$

$$= \frac{i}{(r-1)!} e^{iw_n x} \sum_{s=0}^{r-1} \left[\binom{r-1}{s} (ix)^s \right.$$

$$\left. \int_{-\alpha}^{\alpha} f_{-1}(y)(-iy)^{(r-1-s)} e^{-iw_n y}\, dy \right]$$

$$= \frac{i}{(r-1)!} e^{iw_r x} \sum_{s=0}^{r-1} \binom{r-1}{s} F_{-1}^{(r-1-s)}(w_n)(ix)^s$$

Thus

$$(f * \gamma_{n,r})(x) = i\, e^{iw_n x} \sum_{s=0}^{r-1} \frac{1}{(r-1-s)!s!} F_{-1}^{(r-1-s)}(w_n)(ix)^s \tag{8.47}$$

for $v_n > 0$ and $x \epsilon T$.

A similar calculation yields

$$(f * \gamma_{n,r})(x) = -i\, e^{iw_n x} \sum_{s=0}^{r-1} \frac{1}{(r-s-1)!s!} F_{1}^{(r-1-s)}(w_n)(ix)^s \tag{8.48}$$

for $v_n < 0$ and $x \epsilon T$.

Now let

$$A = \{n:\ 1 \le n \le N \text{ and } v_n > 0\}$$

and

$$B = \{n:\ 1 \le n \le N \text{ and } v_n < 0\}$$

From 8.46), 8.47) .48), we obtain, for $x \epsilon T$,

$$0 = \sum_{n=1}^{N} \sum_{r=1}^{\beta_n} c_{n,r}\, (f * \gamma_{n,r})(x)$$

$$= i \left\{ \sum_{n \epsilon A} e^{iw_n x} \sum_{r=1}^{\beta_n} c_{n,r} \sum_{s=0}^{r-1} \frac{1}{(r-1-s)!s!} F_{-1}^{(r-1-s)}(w_n)(ix)^s \right.$$

$$\left. - \sum_{n \epsilon B} e^{iw_n x} \sum_{r=1}^{\beta_n} c_{n,r} \sum_{s=0}^{r-1} \frac{1}{(r-1-s)!s!} F_{1}^{(r-1-s)}(w_n)(ix)^s \right\}$$

$$= i \left\{ \sum_{n \epsilon A} e^{iw_n x} \sum_{s=0}^{\beta_n - 1} \frac{i^s}{s!} \left[\sum_{r=s+1}^{\beta_n} \frac{c_{n,r}}{(r-1-s)!} F_{-1}^{(r-1-s)}(w_n) \right] x^s \right.$$

$$\left. - \sum_{n \epsilon B} e^{iw_n x} \sum_{s=0}^{\beta_n - 1} \frac{i^s}{s!} \left[\sum_{r=s+1}^{\beta_n} \frac{c_{n,r}}{(r-1-s)!} F_{1}^{(r-1-s)}(w_n) \right] x^s \right\} .$$

For $n \in A$, let

$$(8.49) \qquad d_{n,s} = \frac{i^s}{s!} \sum_{r=s+1}^{\beta_n} \frac{C_{n,r}}{(r - 1 - s)!} F_{-1}^{(r-1-s)}(w_n)$$

and for $n \in B$, let

$$(8.50) \qquad d_{n,s} = -\frac{i^s}{s!} \sum_{r=s+1}^{\beta_n} \frac{C_{n,r}}{(r - 1 - s)!} F_{1}^{(r-1-s)}(w_n).$$

Let

$$\alpha(x) = \sum_{n=1}^{N} e^{iw_n x} \sum_{s=0}^{\beta_n - 1} d_{n,s} x^s.$$

Then, since α is analytic and $\alpha(x) = 0 \ \forall \ x \in T$,

$$(8.51) \qquad \alpha(x) = 0 \ \forall \ x \in \mathfrak{R}.$$

Claim

$$d_{n,s} = 0 \text{ for } 0 \le s \le \beta_n - 1 \text{ and } n = 1, \ldots, N.$$

Proof of claim: Choose an integer t, $1 \le t \le N$, $\ni v_t \le v_n$, $n = 1, \ldots, N$, and if $v_n = v_t$, then $\beta_n \le \beta_t$. Since $\alpha(x) = 0 \ \forall \ x \in \mathfrak{R}$,

$$0 = \lim_{R \to \infty} \frac{1}{R^{\beta_t}} \int_0^R \alpha(x) \, e^{-iw_t x} \, dx$$

$$= \lim_{R \to \infty} \left[\frac{1}{R^{\beta_t}} \int_0^R \left(\sum_{s=0}^{\beta_t - 1} d_{t,s} x^s \right) dx \right.$$

$$\left. + \sum_{\substack{n=1 \\ n \neq t}}^{N} \lim_{R \to \infty} \left[\frac{1}{R^{\beta_t}} \int_0^R \left(\sum_{s=0}^{\beta_n - 1} d_{n,s} x^s \right) e^{i(w_n - w_t)x} \, dx \right] \right.$$

$$= \lim_{R\to\infty} \left[\frac{1}{R^{\beta_t}} \sum_{s=0}^{\beta_t - 1} \frac{1}{s+1} d_{t,s} R^{s+1} \right]$$

$$+ \sum_{\substack{n=1 \\ n\neq t}}^{N} \sum_{s=0}^{\beta_n - 1} d_{n,s} \lim_{R\to\infty} \left[\frac{1}{R^{\beta_t}} \int_0^R x^s e^{i(w_n - w_t)x} \, dx \right]$$

$$= \frac{1}{\beta_t} d_{t,\beta_t - 1} + \sum_{\substack{n=1 \\ n\neq t}}^{N} \sum_{s=0}^{\beta_n - 1} d_{n,s} \lim_{R\to\infty} \left[\frac{1}{R^{\beta_t}} \int_0^R x^s e^{i(w_n - w_t)x} \, dx \right].$$

Thus,

$$d_{t,\beta_t - 1} = -\beta_t \sum_{\substack{n=1 \\ n\neq t}}^{N} \sum_{s=0}^{\beta_n - 1} d_{n,s} \lim_{R\to\infty} \left[\frac{1}{R^{\beta_t}} \rho_{n,s}(R) \right] \tag{8.52}$$

where

$$\rho_{n,s}(R) = \int_0^R x^s e^{i(w_n - w_t)x} \, dx$$

$$= e^{i(u_n - u_t)R} e^{(v_t - v_n)R} \left[\frac{R^s}{i(w_n - w_t)} + \sum_{r=1}^{s} (-1)^r \frac{s!}{(s-r)![i(w_n - w_t)]^{r+1}} R^{s-r} \right] - \frac{(-1)^s}{[i(w_n - w_t)]^{s+1}} .$$

If $v_t < v_n$, then it is clear that

$$\lim_{R\to\infty} \frac{1}{R^{\beta_t}} \rho_{n,s}(R) = 0.$$

If $v_n = v_t$, then $\beta_n \leq \beta_t$, $s \leq \beta_n - 1 \leq \beta_t - 1$ and

$$\left| \frac{1}{R^{\beta_t}} \rho_{n,s}(R) \right| \le \frac{1}{R^{\beta_t}} \left[\frac{R^s}{|w_n - w_t|} + \sum_{r=1}^{s} \frac{s!R^{s-r}}{(s-r)!\,|w_n - w_t|^{r+1}} + \frac{1}{|w_n - w_t|^{s+1}} \right] \xrightarrow[R \to \infty]{} 0.$$

Therefore 8.53) holds $\forall\ n \neq t$. It then follows from 8.52) that $a_{t,\beta_t - 1} = 0$. Now repeat the above argument (starting from "Proof of claim") with β_t replaced by $\beta_t - 1$. By continuing this procedure, we eventually obtain

8.54) $\qquad a_{n,s} = 0$ for $0 \le s \le \beta_n - 1$ and $n = 1, \ldots, N$.

This completes proof of claim.

Now suppose $n \in A$. Then by 8.49) and 8.54),

$$\sum_{r=s+1}^{\beta_n} \frac{c_{n,r}}{(r-1-s)!} F_{-1}^{(r-1-s)}(w_n) = 0 \text{ for } 0 \le s \le \beta_n - 1.$$

Let a matrix C be defined by

$$C = \begin{bmatrix} c_{n,\beta_n} & & & & & \\ c_{n,\beta_n - 1} & c_{n,\beta_n} & & & 0 & \\ c_{n,\beta_n - 2} & c_{n,\beta_n - 1} & \frac{1}{2!} c_{n,\beta_n} & & & \\ c_{n,\beta_n - 3} & c_{n,\beta_n - 2} & \frac{1}{2!} c_{n,\beta_n - 1} & \frac{1}{3!} c_{n,\beta_n} & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ \cdot & & & & & & \cdot \\ c_{n,1} & c_{n,2} & \frac{1}{2!} c_{n,3} & \frac{1}{3!} c_{n,4} & \cdots & \frac{1}{(\beta_n - 1)!} c_{n,\beta_n} \end{bmatrix}$$

We also define the vertical vector

$$V = \begin{bmatrix} F_{-1}(w_n) \\ F_{-1}^{(1)}(w_n) \\ F_{-1}^{(2)}(w_n) \\ \cdot \\ \cdot \\ \cdot \\ F_{-1}^{(\beta_n-1)}(w_n) \end{bmatrix}$$

Then the system of equations 8.55) may be written as

8.56) $$C \cdot V = 0.$$

We wish to conclude that $V = 0$. It suffices to show that

8.57) $$\det(C) \neq 0.$$

We have

$$\det(C) = \left(\sum_{s=0}^{\beta_n-1} \frac{1}{s!} \right) \left(C_{n,\beta_n} \right)^{\beta_n}.$$

By 8.41), $C_{n,\beta_n} = 0$ and therefore 8.57) holds and $V = 0$. That is,

8.58) $$F_{-1}^{(s)}(w_n) = 0 \text{ for } s = 0, \ldots, \beta_n - 1$$

and $n \in A$.

Furthermore, since $\beta_n \leq \eta_F(w_n)$,

$$F^{(s)}(w_n) = 0 \text{ for } s = 0, \ldots, \beta_n - 1$$

and $n \in A$.

Since $F^{(s)}(w_n) = F_{-1}^{(s)}(w_n) + F_1^{(s)}(w_n)$, we obtain

8.59) $$F_1^{(s)}(w_n) = 0 \text{ for } s = 0, \ldots, \beta_n - 1$$

and $n \epsilon A$.

From 8.58) and 8.59), we conclude that for $n \epsilon A$, the w_n are zeroes of both F_{-1} and F_1 of orders $\geq \beta_n$.

If $n \epsilon B$, a similar argument yields the same result. Therefore all the w_n, $n = 1, \ldots, N$, are zeroes of both F_{-1} and F_1 of orders $\geq \beta_n$.

We have

$$G(w) = F(w) \prod_{n=1}^{N} \left[\frac{1 - \frac{w}{\bar{w}_n}}{1 - \frac{w}{w_n}} \right]^{\beta_n} .$$

Define

$$G_r(w) = F_r(w) \prod_{n=1}^{N} \left[\frac{1 - \frac{w}{\bar{w}_n}}{1 - \frac{w}{w_n}} \right]^{\beta_n} , \quad r = -1, 1.$$

Then $G_r(u)$, $u \epsilon \mathfrak{R}$, is square-summable. Let g_r be the inverse transform of G_r, $r = -1, 1$. We have

$$F = F_{-1} + F_1 .$$

Therefore,

$$G = G_{-1} + G_1$$

and

$$g = g_{-1} + g_1 .$$

Now

$$I(f_{-1}) = [a_f, s_m]$$

and

$$I(f_1) = [t_m, b_f].$$

By Theorem 7,

$$I(g_r) = I(f_r), \; r = -1, 1.$$

Therefore,

$$Q_k = P_m.$$

Also, by (if necessary) adjusting the values of g_{-1} and g_1 on sets of measure zero,

$$g_{-1}(x) = \begin{cases} g(x) \text{ for } a_f \le x \le s_m \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$g_1(x) = \begin{cases} g(x) \text{ for } t_m \le x \le b_f \\ 0 \quad \text{otherwise} \end{cases}$$

Q.E.D.

Proof of Theorem 11: Let $R(F) = \{w\text{: } \operatorname{Im} w > 0 \text{ and either } w \in W \text{ or } \overline{w} \in W\} = \{w_n\text{: } n = 1, \ldots, N\}$. Let

$$A = D(F) \setminus R(F) = \{z_n\text{: } n = 1, \ldots, \infty\} \text{ with } |z_n| \le |z_{n+1}|,$$
$$n = 1, \ldots, \infty.$$

In the factorization for F given in Theorem 5, a finite number of the factors in the infinite product may be reordered without affecting the value of the product. Therefore we have

$$F(w) = \frac{1}{r!} F^{(r)}(0) w^r \prod_{n=1}^{N} \left[\left(1 - \frac{w}{w_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(\bar{w}_n)} \right]$$

$$\cdot \prod_{n=1}^{\alpha} \left[\left(1 - \frac{w}{z_n}\right)^{n_F(z_n)} \left(1 - \frac{w}{\bar{z}_n}\right)^{n_F(\bar{z}_n)} \right]$$

where $r = 2n_F(0)$.

For $z_n \in A$, $n_F(z_n) = n_F(\bar{z}_n)$ and

$$F^*(w) = \frac{1}{r!} \overline{F^{(r)}(0)}\, w^r \prod_{n=1}^{N} \left[\left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{w_n}\right)^{n_F(\bar{w}_n)} \right]$$

$$\cdot \prod_{n=1}^{\alpha} \left[\left(1 - \frac{w}{z_n}\right)^{n_F(z_n)} \left(1 - \frac{w}{\bar{z}_n}\right)^{n_F(\bar{z}_n)} \right].$$

Now $\sigma_F(w) \leq n_F(w)\ \forall\, w \in \mathbb{C}$ and

$$\left[\prod_{z \in W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)} \right] \prod_{n=1}^{N} \left[\left(1 - \frac{w}{w_n}\right)^{n_F(w_n)} \left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(\bar{w}_m)} \right]$$

$$= \prod_{n=1}^{N} \left[\left(1 - \frac{w}{\bar{w}_n}\right)^{n_F(w)} \left(1 - \frac{w}{w_n}\right)^{n_F(\bar{w}_n)} \right].$$

Let

$$G(w) = F(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)} .$$

and let g be its inverse transform. Then

$$F^{*}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} G(w)$$

and

$$\tilde{f}(x) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} g(x) \text{ a.e.}$$

Therefore $S(g) = S(\tilde{f})$ and

$$S(g)' = S(\tilde{f})' = (-\infty, -b_f) \cup \left[\bigcup_{m=1}^{M} (-P_m)\right] \cup (-a_f, \infty).$$

Since $0 \varepsilon P_{m_0}$, $0 \varepsilon - P_{m_0}$ and

$$P_{m_0} \cap (-P_{m_0}) \neq \phi.$$

Therefore, by Theorem 10,

$$P_{m_0} = -P_{m_0}$$

Let

$$g_{-1}(x) = \begin{cases} g(x) & \text{for } x < 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_1(x) = \begin{cases} g(x) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\tilde{f}_{-1}(x) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} g_1(x) \text{ a.e.}$$

and

$$F^*_{-1}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} G_1(w).$$

By Theorem 10, each $z \epsilon W(F)$ is a zero of F_1 of order $\geq \sigma_F(z)$ and

$$G_1(w) = F_1(w) \prod_{z \epsilon W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}}\right]^{\sigma_F(z)}$$

Therefore

$$F^*_{-1}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} F_1(w) \prod_{z \epsilon W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}}\right]^{\sigma_F(z)} \qquad \text{Q.E.D.}$$

Proof of Theorem 12: Let $R(F)$, A, G and g be defined as in the proof of Theorem 11. Then

$$G(w) = F(w) \prod_{z \epsilon W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}}\right]^{\sigma_F(z)} ,$$

$$F^*(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} G(w),$$

$$\tilde{f}(x) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} g(x) \text{ a.e.}$$

and

$$S(g)' = (-\infty, -b_f) \cup \left[\bigcup_{m=1}^{M} (-P_m)\right] \cup (-a_f, \infty).$$

Since $P_{m_1} \cap (-P_{m_2}) \neq \emptyset$, it follows by Theorem 10 that $P_{m_1} = -P_{m_2}$. Therefore, f and g are both zero a.e. on $(-t_{m_2}, -s_{m_2}) \cup (s_{m_2}, t_{m_2})$.

Let

$$g_{-1}(x) = \begin{cases} g(x) \text{ for } x \leq -t_{m_2} \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$g_2(x) = \begin{cases} g(x) \text{ for } x \geq - s_{m_2} \\ 0 \quad \text{otherwise} \end{cases}$$

Let $f_2(x) = f_0(x) + f_1(x)$. Then

$$f(x) = f_{-1}(x) + f_2(x).$$

By Theorem 10, each $z \epsilon W(F)$ is a zero of F_{-1} and F_2 of orders $\geq \sigma_F(z)$,

$$G_{-1}(w) = F_{-1}(w) \prod_{z \epsilon W(F)} \left[\frac{1 - \frac{w}{z}}{1 - \frac{w}{\bar{z}}}\right]^{\sigma_F(z)}$$

$$G_2(w) = F_2(w) \prod_{z \epsilon W(F)} \left[\frac{1 - \frac{w}{z}}{1 - \frac{w}{\bar{z}}}\right]^{\sigma_F(z)}$$

Now let

$$g_0(x) = \begin{cases} g(x) \text{ for } -s_{m_2} \leq x \leq s_{m_2} \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$g_1(x) = \begin{cases} g(x) & \text{for } x \geq t_{m_2} \\ \\ 0 & \text{otherwise} \end{cases}$$

Then $g(x) = g_0(z) + g_1(x)$. Again by Theorem 10, each $z \in W(F)$ is a zero of F_0 and F_1 of orders $\geq \sigma_F(z)$,

$$G_0(w) = F_0(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)} \tag{6.60}$$

and

$$G_1(w) = F_1(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{\bar{z}}}{1 - \frac{w}{z}} \right]^{\sigma_F(z)} \tag{6.61}$$

Since

$$\tilde{f}(x) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} g(x) \text{ a.e.,}$$

it follows that

$$\tilde{f}_{-1}(x) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} g_1(x) \text{ a.e.}$$

and

$$\tilde{f}_0(x) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} g_0(x) \text{ a.e.}$$

Therefore

8.62) $$F^*_{-1}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} G_1(w)$$

and

8.63) $$F^*_0(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} G_0(w).$$

Puting 8.60), 8.61), 8.62), and 8.63) together, we obtain

$$F^*_{-1}(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} F_1(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{z}}{1 - \frac{w}{\bar{z}}}\right]^{\sigma_F(z)}$$

and

$$F^*_0(w) = \frac{\overline{F^{(r)}(0)}}{F^{(r)}(0)} F_0(w) \prod_{z \in W(F)} \left[\frac{1 - \frac{w}{z}}{1 - \frac{w}{\bar{z}}}\right]^{\sigma_F(z)} \qquad \text{Q.E.D.}$$

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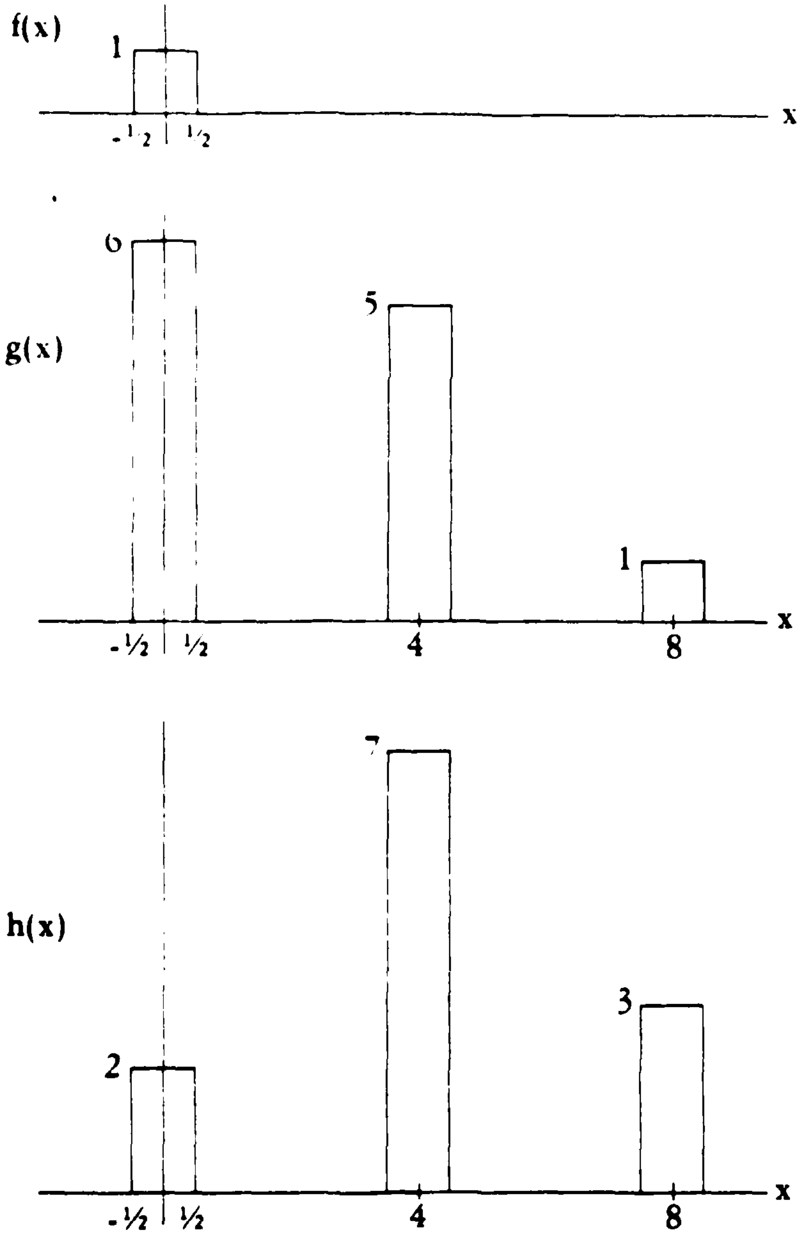


Figure 1.

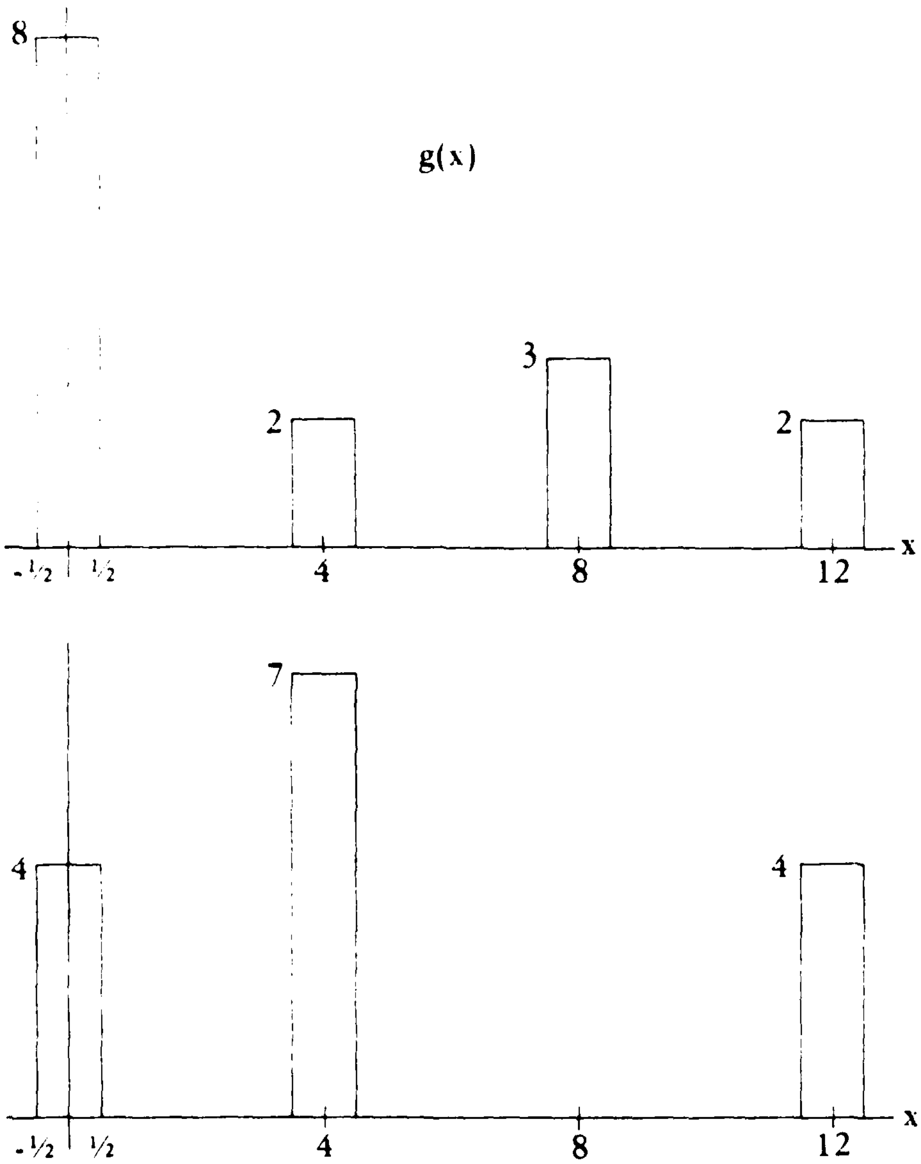


Figure 2.

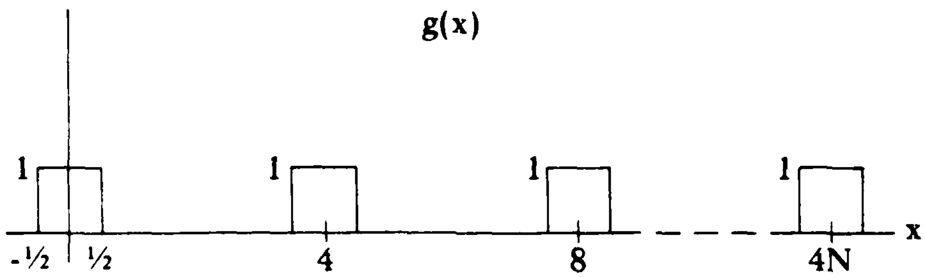


Figure 3.

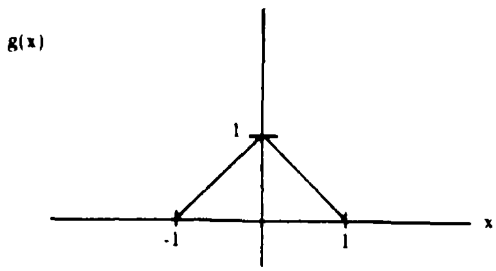
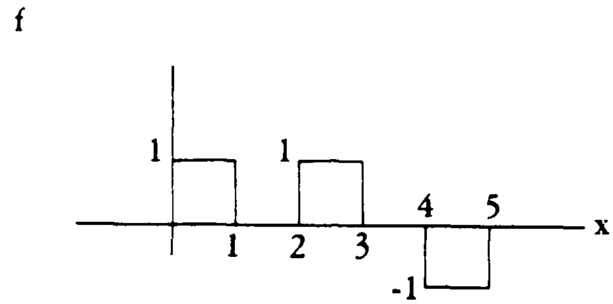


Figure 4.



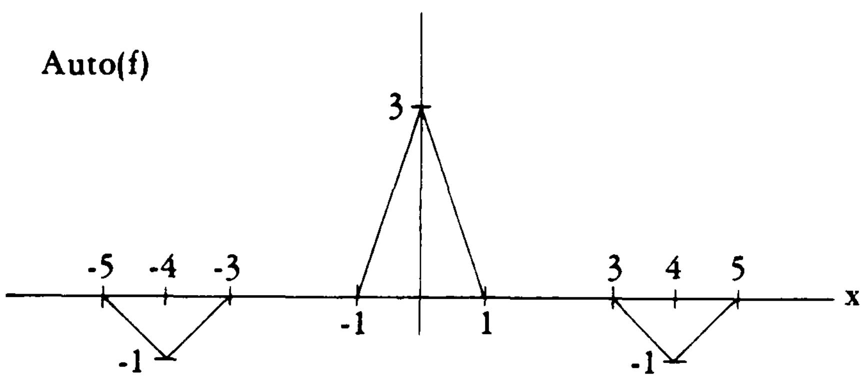


Figure 5.

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APPENDIX C

COMMENTS ON CLAIMS CONCERNING THE UNIQUENESS OF SOLUTIONS TO THE PHASE RETRIEVAL PROBLEM

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Abstract

Questions are raised concerning some claims by A.H. Greenaway and R.H.T. Bates concerning the uniqueness of solutions to the phase retrieval problem for functions with disconnected support. A counterexample is presented.

January 1981

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INTRODUCTION

In this letter, questions are raised concerning some claims by Greenaway [1] and Bates [2]. These papers are concerned with the question of uniqueness of solutions to the phase retrieval problem. This problem, in the one-dimensional case, can be stated as follows.

Let f be a complex-valued function on the real line which vanishes outside of some finite interval. Let F be its Fourier transform. Given the modulus of F on the real line, i.e., $|F(u)|$ for all real u, the problem is to reconstruct the original function, f, from this information. The general uniqueness question is: *How many other functions, $g \neq f$, exist which vanish outside of some finite interval and whose Fourier transforms satisfy $|G(u)| = |F(u)|$ for all real u?*

2. GREENAWAY'S PAPER [1]

Greenaway considers a situation in which the unknown function, f, is known to be zero outside of the union of two disjoint intervals (a,c) and (d,b). In other words

$$f = g + h \ .$$

where g is zero outside of (a,c) and h is zero outside of (d,b) (see Figure 1).

Now let F, G and H be the Fourier transforms of f, g and h, respecively, extended by analyticity into the complex plane:

$$F(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

where $w = u+iv$ and u and v are real. The modulus of F on the real line, i.e., $|F(u)|$, is given.

The question is: *To what extent do the conditions described above determine the function f?*

The functions

$$e^{i\alpha}f(x+\beta) \quad \text{and} \quad e^{i\alpha}\overline{f(-x+\beta)},$$

where the overbar denotes complex conjugation and α and β are real, have the same Fourier modulus on the real line as does f. If any of these functions are also zero outside of the union of the intervals (a,c) and (d,b), then they satisfy all the requirements and qualify as alternate solutions. These solutions will be said to be *associated* with the solution f.

Now the revised question is: *Are there any other solutions not associated with f?*

Let w_0 be a non-real zero of F, and let

$$F_1(w) = F(w)\,\frac{w-\overline{w_0}}{w-w_0} \quad .$$

The function F_1 can be viewed as being gotten from F by first removing a zero at w_0 and then adding a zero at $\overline{w_0}$. In other words, the zero at w_0 has been "flipped" about the real line. Now for real w, $w=u$,

$$\left|\frac{u-\overline{w_0}}{u-w_0}\right| = 1$$

and therefore

$$|F_1(u)| = |F(u)| \text{ for all real } u.$$

Hofstetter [3] and Walther [4] proved that if f_1 is any function which vanishes outside of some finite interval and $|F_1(u)| = |F(u)|$ for all real u, then F_1 is gotten from F by flipping various sets of non-real zeroes of F and multiplying by a constant of modulus 1 and by an exponential function. In particular, if F_1 is obtained from F by flipping the set of all its non-real zeroes, then its inverse transform, f_1, satisfies

$$f_1(x) = \overline{f(-x)},$$

and thus, if f_1 vanishes outside the union of (a,c) and (d,b), then f_1 is a solution associated with f. (Here, if a zero of F has multiplicity n, it must be flipped n times.)

Now let Z(F) denote the set of all non-real zeroes of F.

Greenaway claims that if F_1 is obtained from F by flipping any proper subset, S, of Z(F) (i.e., $S \neq Z(F)$) and if f_1 vanishes outside of the union of (a,c) and (d,b), then all the points in S are zeroes of both G and H.

Thus, if G and H have no zeroes in common (which would usally be the case if g and h are gotten more or less randomly from the real world), then it would follow that the only solutions are f and its associated solutions.

Greenaway's claim is true in the special case in which F has only a finite number of non-real zeroes. (Actually, Greenaway's proof holds

only for the more restricted case in which F has a finite number of non-real zeroes of order 1. However, the case of higher order zeroes can be taken care of by an extension of his argument. See [5].)

The following counterexample shows that Greenaway's claim is not true in general. In this counterexample, the set Z(F) is infinite and S is an infinite proper subset of Z(F).

Counterexample:

Let

$$\phi(x) = \begin{cases} 1-|x| & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1 . \end{cases}$$

See Figure 2. Then the Fourier transform, Φ, of ϕ is given by

$$\Phi(w) = \frac{\sin^2\left(\frac{w}{2}\right)}{\left(\frac{w}{2}\right)^2} = \operatorname{sinc}^2\left(\frac{w}{2}\right)$$

Note that Φ has no non-real zeroes.

Now let

$$g(x) = 8\phi(x)$$

and

$$h(x) = 2\phi(x-4) + 3\phi(x-8) + 2\phi(x-12).$$

Then $G=8\Phi$ has no non-real zeroes and hence G and H have no non-real zeroes in common. Let

$$f(x) = g(x) + h(x)$$

$$= 8\phi(x) + 2\phi(x-4) + 3\phi(x-8) + 2\phi(x-12)$$

and let

$$a = -1, \quad c = 1, \quad d = 3, \quad b = 13.$$

See Figure 3. Then $a < c < d < b$, the intervals (a,c) and (d,b) are disjoint, and f is zero outside the union of (a,c) and (d,b). The Fourier transform of f is

$$F(w) = \left(8+2e^{-4iw} + 3e^{-8iw} + 2e^{-12iw}\right)\Phi(w)$$

$$= 2\left(e^{-4iw} + 2\right)\left(e^{-8iw} - .5e^{-4iw} + 2\right)\Phi(w). \qquad (1)$$

Now let

$$g_1(x) = 4\phi(x)$$

and

$$h_1(x) = 7\phi(x-4) + 4\phi(x-12).$$

Then $G_1=4\Phi$ has no non-real zeroes and hence G_1 and H_1 have no non-real zeroes in common. Let

$$f_1(x) = g_1(x) + h_1(x)$$

$$= 4\phi(x) + 7\phi(x-4) + 4\phi(x-12).$$

See Figure 3. Then f_1 is also zero outside of the union of (a,c) and (d,b). The Fourier transform of f_1 is

$$F_1(w) = \left(4 + 7e^{-4iw} + 4e^{-12iw}\right) \Phi(w)$$

$$= 2\left(2e^{-4iw} + 1\right)\left(e^{-8iw} - .5e^{-4iw} + 2\right) \Phi(w)$$

$$= 2e^{-4iw} \left(e^{4iw} + 2\right) \left(e^{-8iw} - .5e^{-4iw} + 2\right) \Phi(w) \tag{2}$$

It follows from (1) and (2) that

$$F_1(w) = e^{-4iw} \left(\frac{e^{4iw}+2}{e^{-4iw}+2} \right) F(w).$$

Now, for real w,w=u,

$$\left| e^{-4iu} \frac{e^{4iu}+2}{e^{-4iu}+2} \right| = 1.$$

Therefore

$$|F_1(u)| = |F(u)| \quad \text{for all real } u.$$

Thus f and f_1 are both solutions and it is clear that they are not associated.

In order to see which zeroes must be flipped to get F_1 from F, let

$$\Gamma_1(w) = e^{-4iw} + 2$$

and

$$\Gamma_2(w) = e^{-8iw} - .5e^{-4iw} + 2.$$

Then

$$F(w) = 2\Gamma_1(w)\ \Gamma_2(w)\Phi(w) \tag{3}$$

and

$$F_1(w) = 2e^{-4iw}\ \overline{\Gamma_1(\overline{w})}\ \Gamma_2(w)\Phi(w) \tag{4}$$

Since Φ has no non-real zeroes and e^{-4iw} is never zero, it follows from (3) and (4) that

$$Z(F) = Z(\Gamma_1) \cup Z(\Gamma_2)$$

and

$$Z(F_1) = \overline{Z(\Gamma_1)} \cup Z(\Gamma_2)$$

where

$$\overline{Z(\Gamma_1)} = \left\{ \overline{w} : w \in Z(\Gamma_1) \right\} .$$

Thus the zeroes of F which are in $S = Z(\Gamma_1)$ are flipped. The sets $Z(\Gamma_1)$ and $Z(\Gamma_2)$ are given by

$$Z(\Gamma_1) = \left\{ \frac{\pi}{4} + \frac{\pi}{2} n + \frac{i}{4} \log 2 : n = 0, \pm 1, \pm 2, \ldots \right\}$$

and

$$Z(\Gamma_2) = \left\{ \pm \frac{1}{4} \tan^{-1} \sqrt{31} + \frac{\pi}{2} n + i \frac{1}{8} \log 2 : n = 0, \pm 1, \pm 2, \ldots \right\} .$$

See Figure 4. The flipping of the zeroes in S is followed by multiplication by the exponential e^{-4iw}. The latter simply has the effect of translating f_1 into the proper position.

In the above example the function φ could be replaced by any function which is zero outside of the interval $(-1,1)$ and whose Fourier transform has no non-real zeroes. For example, φ could be replaced by

$$\varphi_1(x) = (\varphi*\varphi)(2x),$$

where * denotes convolution, or by

$$\varphi_2(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

3. BATE'S PAPER [2]

Bates considers the situation in which

$$f(x) = \sum_{n=1}^{N} f_n(x)$$

where each f_n is zero outside of an interval I_n and the intervals I_n, $n=1\ldots N$ are pairwise disjoint. He claims that if the Fourier transforms, F_n, have no non-real zeros common to all of them, then f and its associated solutions are the only functions with compact support and whose Fourier transforms have the same moduli as that of the Fourier transform of f. Thus Bates claims even more than Greenaway does. Therefore, the above example is also a counterexample to Bates' claim.

A stronger separation condition on the I_n which does work can be found in [5].

Acknowledgements

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Figure Captions

Figure 1: Member of the class of functions with disconnected support. Note: Although the functions g and h are represented here as positive real functions, they can be complex-valued.

Figure 3: Functions $f(x)$ (above) and $f_1(x)$ (below) have the same Fourier modulus.

Figure 4: Above: non-real zerores of F. Below: non-real zeroes of F_1. The circled zeroes are flipped.

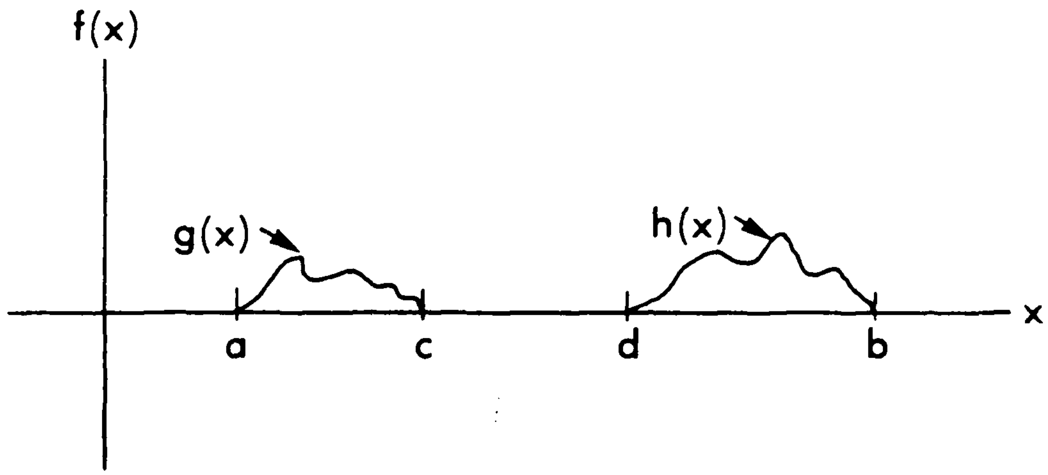

Figure 1

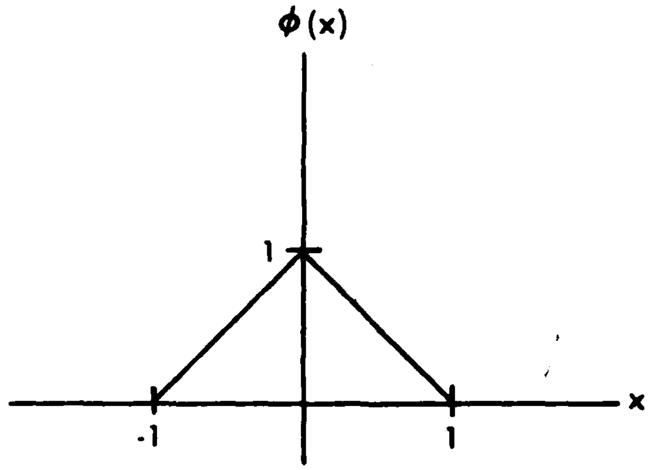

Figure 2

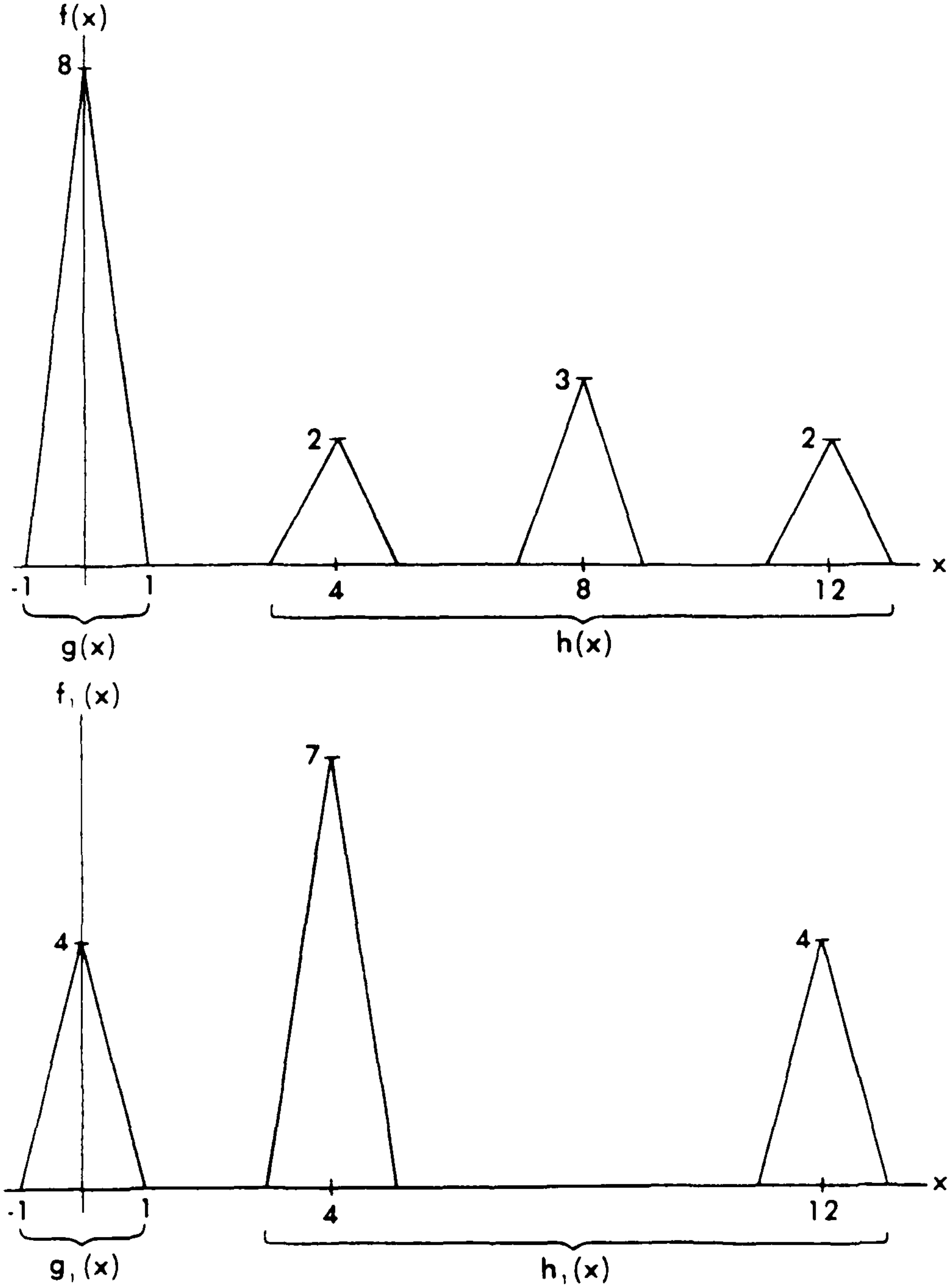

Figure 3

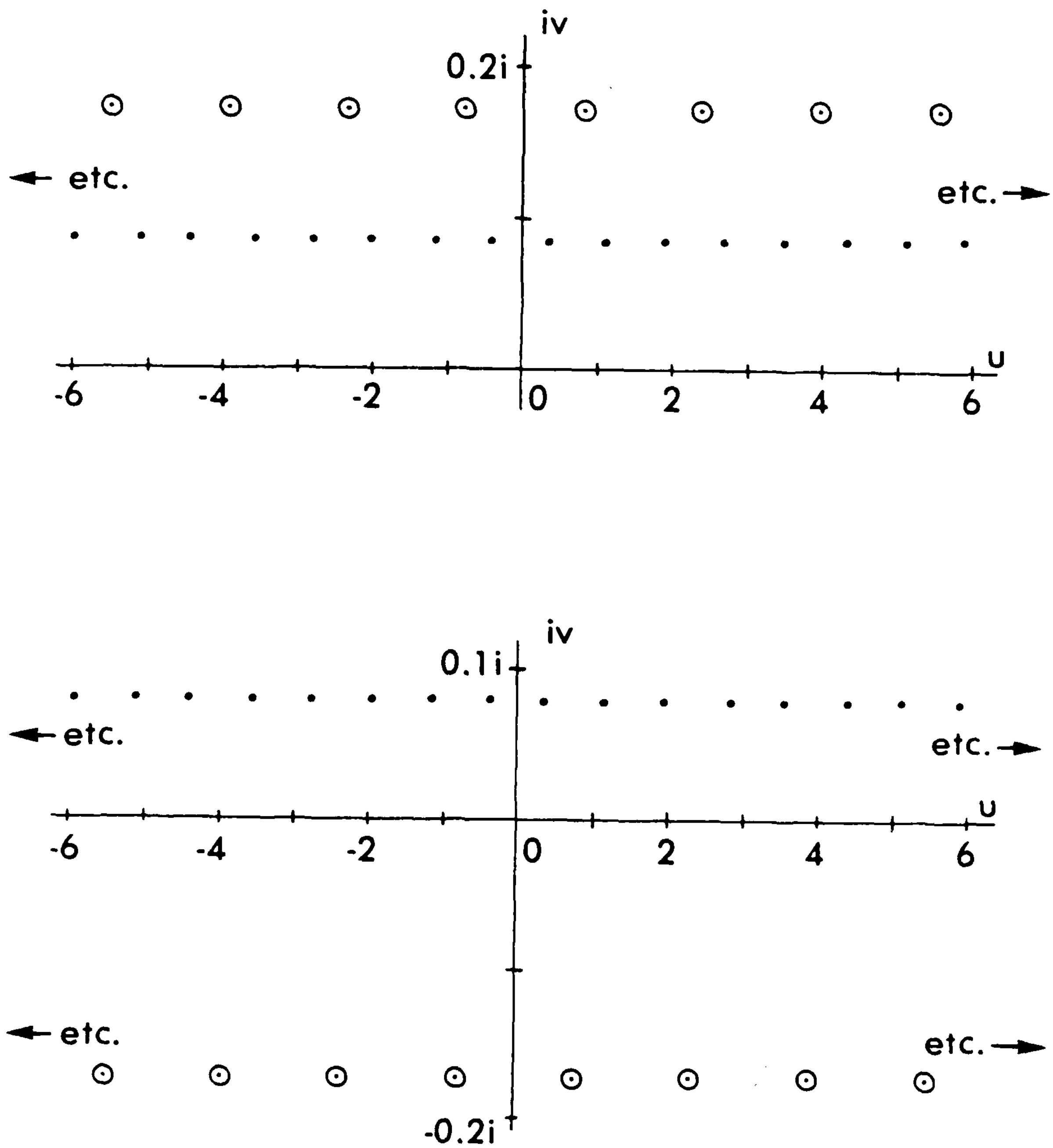

Figure 4

APPENDIX D

WB6. Determining the Support of an Object from the Support of Its Autocorrelation.* J. R. FIENUP AND T. R. CRIMMINS, *Radar and Optics Division, Environmental Research Institute of Michigan, P.O. Box 8618, Ann Arbor, Michigan 48107.* —In astronomy, x-ray crystallography, and other disciplines, one often wishes to reconstruct an object distribution from its autocorrelation or, equivalently, from the modulus of its Fourier transform (i.e., the phase retrieval problem). It is also useful to be able to reconstruct just the support of the object (i.e., the region on which it is nonzero). In some cases, for example, to find the relative locations of a number of pointlike stars, the object's support is the desired information. In addition, once the object's support is known, the reconstruction of the object distribution by the iterative method[1] is simplified. We show several methods of finding sets which contain the support of an object, based on the support of its autocorrelation. The smaller these sets are, the more information they give about the support of the object. Particularly small sets containing the object's support are given by intersections of its autocorrelation's support with translates of its autocorrelation's support. It will be shown that for special cases this gives rise to a unique reconstruction of the support of the object from the support of its autocorrelation. (13 min.)

* Work support by AFOSR

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Presented at the 1980 Annual Meeting of the Optical Society of America, Chicago, Illinois, 15 October 1980; Abstract: J. Opt. Soc. Am. 70, 1581 (1980).

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